ON THE INTEGRAL SOLUTIONS OF TRIPLE EQUATIONS $x+y=a^2, 2x+y=a^2+b^2, x+2y=a^2+5c^2$

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Abstract: *An attempt is made to obtain non-zero distinct integer quintuples* (x, y, a, b, c) *satisfying the system of three equations* $x+y=a^2, 2x+y=a^2+b^2, x+2y=a^2+5c^2$. *Different sets of integer solutions are presented.*

Keywords: **system of triple equations, triple equations with five unknowns, integer solutions.**

1. INTRODUCTION

In [1], an attempt has been made to obtain pairs of non-zero distinct integers x, y such that, in each pair

i. $x+y=a^2, 2x+y=b^2, x+2y=a^3$

ii. $x+y=a^2, 2x+y=b^2, x+2y=c^3$

In [2], Illustrates the analysis of obtaining different sets of distinct integer solutions to two systems of triple equations with five unknowns given by

i. $x+y=a^2, 2x+y=b^2, x+2y=3c^2$

ii. $x+y=a^2, 2x+y=b^2, x+2y=2c^2$ Respectively.

In [3], the system of Triple Equations with five variables $x+y=a^2, 2x+y=b^2, x+2y=a^2-c^2$ is studied for its integer solutions.
This communication exhibits different sets of non-zero distinct integer solutions for the system of Triple Equations with five unknowns given by $x+y=a^2, 2x+y=a^2+b^2, x+2y=a^2+5c^2$.

Notations:

1) Regular polygonal number of rank n with sides m

$$t_{m,n}=n\left(1+\frac{(n-1)(m-2)}{2}\right)$$

2) Pronic number of rank n

$$pr_n=n(n+1)$$

3) Pyramidal number of rank n with sides m

$$p_n^m = \frac{1}{6}[n(n+1)][(m-2)n+(5-m)]$$

4) Pentatope number of rank n

$$pt_n = \frac{n(n+1)(n+2)(n+3)}{24}$$

2. Method of analysis

Consider the system of equations

$$x + y = a^2 \tag{1}$$

$$2x + y = a^2 + b^2 \tag{2}$$

$$x + 2y = a^2 + 5c^2 \tag{3}$$

Eliminating x and y between (1) to (3), the resulting equation is

$$a^2 = b^2 + 5c^2 \tag{4}$$

which is satisfied by

$$a = 5r^2 + s^2, b = 5r^2 - s^2, c = 2rs \tag{5}$$

Now, (2)-(1) $\Rightarrow$ $x = b^2$ (6)

and (1) $\Rightarrow y = a^2 - x = a^2 - b^2$ (7)

Using (5) in (6) and (7), the values of x and y satisfying the system of equations (1) to (3), are given by

$$x = 25r^4 - 10r^2s^2 + s^4,\ y = 20r^2s^2 \tag{8}$$

In addition to the above solutions (8) of (1) to (3), there are other choices of solutions to the system of equations under consideration that are illustrated below:

Choice 1:

Write (4) as

$$b^2 + 5c^2 = a^2 * 1 \tag{9}$$

Assume $a = 9(\alpha^2 + 5\beta^2)$ (10)

And $1 = \frac{(2+i\sqrt{5})(2-i\sqrt{5})}{9}$ (11)

Substituting (10) and (11) in (9), and applying the method of factorization, define

$$b + i\sqrt{5}c = \frac{(2+i\sqrt{5})}{3} * 3^2(\alpha + i\sqrt{5}\beta)^2$$

Equating the rational and irrational parts in the above equation, one obtains

$$b = 3(2\alpha^2 - 10\beta^2 - 10\alpha\beta) \tag{12}$$

$$c = 3(\alpha^2 - 5\beta^2 + 4\alpha\beta) \tag{13}$$

Using the values of a, b, c given by (10), (12) and (13) in (6) and (7), the required values of x and y satisfying (1) to (3), are given by

$$x = 36\alpha^4 + 900\beta^4 + 540\alpha^2\beta^2 + 1800\alpha\beta^3 - 360\alpha^3\beta$$

$$y = 45\alpha^4 + 1125\beta^4 + 270\alpha^2\beta^2 - 1800\alpha\beta^3 + 360\alpha^3\beta$$

A few numerical values are presented below in Table 1:

Table 1: Numerical values

α	β	a	b	c	x	y
1	2	189	-174	-33	30276	5445
2	1	81	-66	21	4356	2205

Properties:

1) $x(\alpha,1)-72p_{\alpha^2}^2+936t_{4,\alpha}+720p_{\alpha}^5-3600p_{\alpha}^2$ *is a perfect square*

2) $81\left[y(\alpha,1)-45(t_{4,\alpha})^2-720p_{\alpha}^5+3600p_{\alpha}^2-1710t_{4,\alpha}\right]$ is a cubical integer.

Choice 2:

In (11), 1 can also be written as

$$1=\frac{(1+i4\sqrt{5})(1-i4\sqrt{5})}{81} \quad (14)$$

Assume $a=81(\alpha^2+5\beta^2)$ (14a)

Substituting (14) and (14a) in (9), and applying the method of factorization, define

$$b+i\sqrt{5}c=\frac{(1+i4\sqrt{5})}{9}*9^2(\alpha+i\sqrt{5}\beta)^2$$

Equating the rational and irrational parts in the above equation, one obtains

$$b=9(\alpha^2-5\beta^2-40\alpha\beta) \quad (15)$$

$$c=9(4\alpha^2-20\beta^2+2\alpha\beta) \quad (16)$$

Using the values of a,b,c given by (14a) and (15) and (16) in (6) and (7), the required values of x and y satisfying (1) to (3), are obtained as

$$x=(81\alpha^4+2025\beta^4-6480\alpha^3\beta+32400\alpha\beta^3+128790\alpha^2\beta^2)$$

$$y=6480\alpha^4+162000\beta^4-63180\alpha^2\beta^2+6480\alpha^3\beta-32400\alpha\beta^3$$

A few numerical values are presented below in Table 2:

Table 2: Numerical values

α	β	a	b	c	x	y
1	1	486	-396	-126	156816	79380
1	2	1701	-891	-648	793881	2099520

Properties:

1) $\left[x(\alpha,1)-81(t_{4,\alpha})^2+12960p_{\alpha}^5-64800p_{\alpha}^2-102870t_{4,\alpha}\right]$ *is a perfact square*

2) $5\left[y(\alpha,1)-12960p_{\alpha^2}^2+139320t_{3,\alpha}-12960p_{\alpha}^5-74520t_{3,\alpha}+43740t_{4,\alpha}\right]$ *is a perfact square*

Choice 3:

Express (4) as the system of double equations as in the Table 3 below:

Table 3: system of double equations

System	1	2	3
a+b	c^2	5c	$5c^2$
a-b	5	c	1

Consider system 1:

Solving for a and b, we have

$$a=\frac{c^2+5}{2},\quad b=\frac{c^2-5}{2}$$

Taking $c=2\alpha+1$, we have

$$a = 2\alpha^2 + 2\alpha + 3$$
$$b = 2\alpha^2 + 2\alpha - 2$$

Using the values of a, b, c in (6) & (7), the required values of x and y satisfying (1) to (3), are obtained as

$$x = 4\alpha^4 + 8\alpha^3 - 4\alpha^2 - 8\alpha + 4$$
$$y = 20\alpha^2 + 20\alpha + 5$$

A few numerical values are presented below in Table 4:

Table 4: Numerical values

α	a	b	c	x	y
1	7	2	3	4	45
2	15	10	5	100	125
3	27	22	7	484	245
4	43	38	9	1444	405
5	63	58	11	3364	605

Properties:

1) $6\left[x(\alpha)+16t_{3,\alpha}-4\right]$ *is a nasty number*

2) $y(\alpha)-t_{4^2,\alpha} \equiv 5(\text{mod } 3)$

Observation 1:

From the values of x and y, one may generate second order Ramanujan numbers as well as Pythagorean triples. The process of obtaining the same is illustrated below.

Illustration 1: consider

$$y = 45 = 45\times 1 = 15\times 3 = 9\times 5$$
$$= 23^2 - 22^2 = 9^2 - 6^2 = 7^2 - 2^2$$

Now,

$$23^2 - 22^2 = 9^2 - 6^2 \Rightarrow 23^2 + 6^2 = 22^2 + 9^2 = 565$$
$$23^2 - 22^2 = 7^2 - 2^2 \Rightarrow 23^2 + 2^2 = 22^2 + 7^2 = 533$$
$$9^2 - 6^2 = 7^2 - 2^2 \Rightarrow 9^2 + 2^2 = 6^2 + 7^2 = 85$$

Thus 85,533,565 represent second order ramanujan numbers.

Illustration 2: consider

$$x = 484$$
$$\Rightarrow 242\times 2 = 22\times 22 \Rightarrow 122^2 - 120^2 = 22^2 \Rightarrow 122^2 = 22^2 + 120^2$$

Thus $(22,120,122)$ is a Pythagorean triple.

Illustration 3: consider

$$x = 100 = 50\times 2 = 10\times 10 \Rightarrow 26^2 - 24^2 = 10^2 \Rightarrow 26^2 = 10^2 + 24^2$$

Thus $(10,24,26)$ is a Pythagorean triple.

Illustration 4: consider

$$y = 405 = 405\times1 \quad = 5\times81 \quad = 45\times9 \quad = 135\times3$$
$$= 203^2 - 202^2 = 43^2 - 38^2 = 27^2 - 18^2 = 69^2 - 66^2$$

Now,

$$203^2 - 202^2 = 43^2 - 38^2 \Rightarrow 203^2 + 38^2 = 43^2 + 202^2 = 42653$$
$$203^2 - 202^2 = 27^2 - 18^2 203^2 + 18^2 = 202^2 + 27^2 = 41533$$
$$203^2 - 202^2 = 69^2 - 66^2 \Rightarrow 203^3 + 66^2 = 69^2 + 202^2 = 45565$$
$$43^2 - 38^2 = 27^2 - 18^2 \Rightarrow 43^2 + 18^2 = 27^2 + 38^2 = 2173$$
$$43^2 - 38^2 = 69^2 - 66^2 \Rightarrow 43^2 + 66^2 = 69^2 + 38^2 = 6205$$
$$27^2 - 18^2 = 69^2 - 66^2 \Rightarrow 27^2 + 66^2 = 69^2 + 18^2 = 5085$$

Thus 2173, 5085, 6205, 41533, 42653, 45565 represent second order ramanujan numbers.

Consider system 2:

Solving for a and b, one obtains

$$a = 3c$$
$$b = 2c$$

Using the values of a, b in (6) & (7), the required values of x and y satisfying (1) to (3), are obtained as

$$x = 4c^2$$
$$y = 5c^2$$

A few numerical values are presented below in Table 5:

Table 5: Numerical values

c	a	b	x	y
1	3	2	4	5
2	6	4	16	20
3	9	6	36	45
4	12	8	64	80
5	15	10	100	125

Observation 2:

From the values of x and y, one may generate second order Ramanujan numbers as well as Pythagorean triples. The process of obtaining the same is illustrated below.

Illustration 5: consider

$$y = 125$$
$$= 125\times1 = 25\times5 \Rightarrow 63^2 - 62^2 = 15^2 - 10^2 \Rightarrow 63^2 + 10^2 = 15^2 + 62^2 = 4069$$

Thus 4069 represent second order ramanujan number.

Illustration 6: consider

$$x = 64 = 32\times2 \quad = 16\times4 \quad = 8\times8$$
$$= 17^2 - 15^2 = 10^2 - 6^2 = 8^2$$

Now,

$$17^2 - 15^2 = 10^2 - 6^2 \Rightarrow 17^2 + 6^2 = 10^2 + 15^2 = 325$$

Thus 325 represent second order ramanujan number.

$$17^2 - 15^2 = 8^2 \Rightarrow 17^2 = 8^2 + 15^2$$

Also,

$$10^2 - 6^2 = 8^2 \Rightarrow 10^2 = 8^2 + 6^2$$

Thus $(8,15,17)$ and $(6,8,10)$ are a Pythagorean triples.

Consider system 3:
Solving for a and b, we have

$$a = \frac{5c^2 + 1}{2},\ b = \frac{5c^2 - 1}{2}$$

Taking $c = 2\alpha + 1$, we get

$$a = 10\alpha^2 + 10\alpha + 3$$

$$b = 10\alpha^2 + 10\alpha + 2$$

Using the values of a, b, c in (6) & (7), the required values of x and y satisfying (1) to (3), are obtained as

$$x = 100\alpha^4 + 200\alpha^3 + 140\alpha^2 + 40\alpha + 4$$

$$y = 20\alpha^2 + 20\alpha + 5$$

A few numerical values are presented below in Table 6:

Table 6: Numerical values

α	a	b	c	x	y
1	23	22	3	484	45
2	63	62	5	3844	125
3	123	122	7	14884	245
4	203	202	9	40804	405
5	303	302	11	91204	605

Observation 3:
From the values of x and y, one may generate second order Ramanujan numbers as well as Pythagorean triple. The process of obtaining the same is illustrated below.

Illustration 7: consider

$$y = 245 = 245 \times 1 = 49 \times 5 = 35 \times 7$$
$$= 123^2 - 122^2 = 27^2 - 22^2 = 21^2 - 14^2$$

Now,

$$123^2 - 122^2 = 27^2 - 22^2 \Rightarrow 123^2 + 22^2 = 27^2 + 122^2 = 15613$$
$$123^2 - 122^2 = 21^2 - 14^2 \Rightarrow 123^2 + 14^2 = 21^2 + 122^2 = 15325$$
$$27^2 - 22^2 = 21^2 - 14^2 \Rightarrow 27^2 + 14^2 = 21^2 + 22^2 = 925$$

Thus 15613, 15325, 925 represent second order ramanujan numbers.

Illustration 8: consider

$$y = 605 = 605 \times 1 = 121 \times 5 = 55 \times 11$$
$$= 303^2 - 302^2 = 63^2 - 58^2 = 33^2 - 22^2$$

Now,

$$303^2 - 302^2 = 63^2 - 58^2 \Rightarrow 303^2 + 58^2 = 63^2 + 302^2 = 95173$$
$$303^2 - 302^2 = 33^2 - 22^2 \Rightarrow 303^2 + 22^2 = 33^2 + 302^2 = 92293$$
$$63^2 - 58^2 = 33^2 - 22^2 \Rightarrow 63^2 + 22^2 = 33^2 + 58^2 = 4453$$

Thus 4453, 92293, 95173 represents second order ramanujan numbers.

Illustration 9: consider

$$x = 3844 = 1922 \times 2 = 62 \times 62 \Rightarrow 962^2 - 960^2 = 62^2 \Rightarrow 962^2 = 62^2 + 960^2$$

Thus $(62,960,962)$ is a Pythagorean triples.

CONCLUSION

To conclude, one may search for other sets of integer solutions for the system of triple equations under consideration.

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