On Non-Homogeneous Ternary Bi-Quadratic Equation $x^2 + 7xy + y^2 = z^4$

A. Vijayasankar[1], Sharadha Kumar[2], M.A. Gopalan[3]
[1]*Assistant Professor, Department of Mathematics, National College, Affiliated to Bharathidasan University, Trichy-620 001, Tamil Nadu, India.*
email: avsankar70@yahoo.com
[2]*Research Scholar, Department of Mathematics, National College, Affiliated to Bharathidasan University ,Trichy-620 001, Tamil Nadu, India.*
email: sharadhak12@gmail.com
[3]*Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India.*
email:mayilgopalan@gmail.com

***Abstract**: We obtain infinitely many non-zero integer solutions to the non-homogeneous ternary bi-quadratic equation* $x^2 + 7xy + y^2 = z^4$.

***Keywords:** Ternary bi-quadratic, Integer solutions, Pell equations.*
2010 Mathematics Subject Classification: 11D25, 11D09.

1. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. There is a great interest for mathematicians since antiquity in homogeneous and non-homogeneous bi-quadratic Diophantine equations [1-4]. In this context, one may refer [5-12] for varieties of problems on the bi-quadratic Diophantine equations with three variables. In [13-17], bi-quadratic equation with four unknowns are studied for their integral solutions.

This communication concerns with yet another interesting ternary bi-quadratic equation given by $x^2 + 7xy + y^2 = z^4$ and is analysed for its non-zero distinct integer solutions.

2. METHOD OF ANALYSIS

The ternary bi-quadratic diophantine equation to be solved for its non-zero distinct integral solutions is given by

$$x^2 + 7xy + y^2 = z^4 \tag{1}$$

Introducing the linear transformations

$$x = u + v,\ y = u - v,\ u \neq v \neq 0 \tag{2}$$

in (2), it leads to

$$9u^2 = 5v^2 + z^4 \tag{3}$$

We present below different methods of solving (3) and thus obtain different patterns of integral solutions to (1).

Pattern 1:

It is noted that (3) is satisfied by

$$v = 2rs,\ z^2 = 5r^2 - s^2,\ 3u = 5r^2 + s^2 \tag{4}$$

Taking $r = 3R,\ s = 3S$ (5)

in (5), we have

$$\left.\begin{aligned} u &= 15R^2 + 3S^2 \\ v &= 18RS \end{aligned}\right\} \tag{6}$$

and

$$z^2 = 45R^2 - 9S^2 \tag{7}$$

Now, we have to find R, S and z satisfying (7).

For this, assume

$$R = \alpha^2 + 9\beta^2 \quad (8)$$

Also, 45 can be written as

$$45 = (6+3i)(6-3i) \quad (9)$$

Substituting (8) and (9) in (7) and applying the method of factorization, define

$$(z+3iS) = (6+3i)(\alpha + i3\beta)^2$$

Equating the real and imaginary parts, we get

$$z = 6\alpha^2 - 54\beta^2 - 18\alpha\beta \quad (10)$$

$$S = \alpha^2 - 9\beta^2 + 12\alpha\beta \quad (11)$$

Substituting (8) and (11) in (6), we get

$$\left.\begin{aligned} u &= 18\alpha^4 + 1458\beta^4 + 648\alpha^2\beta^2 + 72\alpha^3\beta - 648\alpha\beta^3 \\ v &= 18\alpha^4 - 1458\beta^4 + 216\alpha^3\beta + 1944\alpha\beta^3 \end{aligned}\right\} \quad (12)$$

Employing (12) in (2), we have

$$\left.\begin{aligned} x &= 36\alpha^4 + 648\alpha^2\beta^2 + 288\alpha^3\beta + 1296\alpha\beta^3 \\ y &= 2916\beta^4 + 648\alpha^2\beta^2 - 144\alpha^3\beta - 2592\alpha\beta^3 \end{aligned}\right\} \quad (13)$$

Thus, (10) and (13) represent non-zero distinct integer solutions to (1).

Note:

It is worth to note that, in addition to (9), one may write 45 as

$$45 = (3+6i)(3-6i)$$

Following the procedure as presented above, the corresponding non-zero distinct integer solutions to (1) are given by

$$x = 63\alpha^4 - 729\beta^4 + 162\alpha^2\beta^2 + 180\alpha^3\beta + 324\alpha\beta^3$$

$$y = -94\alpha^4 + 5103\beta^4 + 162\alpha^2\beta^2 - 36\alpha^3\beta - 1620\alpha\beta^3$$

$$z = 3\alpha^2 - 27\beta^2 - 36\alpha\beta$$

Pattern 2:

Introduction of the linear transformations

$$u = X + 5T,\ v = X + 9T,\ z = 2w \quad (14)$$

in (3) leads to

$$X^2 - 4w^4 = 45T^2 \quad (15)$$

which may be expressed as the system of double equations as presented in Table 1 below:

Table 1: System of double equations

System	1	2	3	4	5
$X+2w^2$	15T	9T	45T	$5T^2$	T^2
$X-2w^2$	3T	5T	T	9	45

Solving each of the above systems, the values of X, T and w are obtained. In view of (2) and (14), the corresponding values of x, y and z for each of the systems in Table 1 are found. Note that the values of x, y and z thus obtained satisfy (1). For the sake of simplicity and brevity, the values of x, y and z satisfying (1), that are obtained through the system of equations in Table 1, are exhibited in Table 2 as follows:

Table 2: Solutions

System	Solutions
1	$x = 96k^2, y = -12k^2, z = 6k$
2	$x = 28k^2, y = -4k^2, z = 2k$
3	$x = 660k^2, y = -44k^2, z = 22k$
4	$x_{n+1} = 20k_{n+1}^2 + 48k_{n+1} + 28$ $y_{n+1} = -(8k_{n+1} + 4)$ $z_{n+1} = 3f_n + \frac{15}{\sqrt{20}} g_n$ where, $k_{n+1} = \frac{1}{10}\left[-5 \pm \left\{\frac{15}{2} f_n + \frac{30}{\sqrt{20}} g_n\right\}\right], n = -1,0,1,....$ $f_n = (9 + 2\sqrt{20})^{n+1} + (9 - 2\sqrt{20})^{n+1}$ $g_n = (9 + 2\sqrt{20})^{n+1} - (9 - 2\sqrt{20})^{n+1}$
5	Set 1: $x = 896, y = -92, z = 22$ Set 2: $x = 252, y = -36, z = 6$ Set 3: $x = 192, y = -28, z = 2$

3.CONCLUSION

In this paper, an attempt has been made to find non-zero distinct integer solutions to the non-homogeneous ternary bi-quadratic equation with three unknowns $x^2 + 7xy + y^2 = z^4$. To conclude, the readers may search for other types of bi-quadratic equations with variables greater than or equal to three to obtain their corresponding solutions.

REFERENCES

[1] *L.J. Mordell, "Diophantine Equations, Academic press", New York, (**1969**).*

[2] *R.D. Carmichael, "The Theory of numbers and Diophantine Analysis, Dover publications", New York, (**1959**).*

[3] *L.E. Dickson, "History of theory of Numbers, Diophantine Analysis", Vol.2, Dover publications, New York, (**2005**).*

[4] *S.G. Telang, "Number Theory", Tata Mc Graw Hill publishing company, New Delhi, (**1996**).*

[5] *M.A. Gopalan, S. Vidhyalakshmi, S. Devibala, "Ternary bi-quadratic Diophantine equation $2^{4n+3}(x^3 - y^3) = z^4$ ", Impact J. Sci. Tech, Vol.4(3), (**2010**),Pp: 57-60.*

[6] *M.A. Gopalan, G. Sangeetha, "Integral solutions of ternary non-homogeneous bi-quadratic equation $x^4 + x^2 + y^2 - y = z^2 + z$ " ,Acta Ciencia Indica, Vol. XXXVIIM, No.4, (**2011**), Pp:799-803.*

[7] *M.A. Gopalan, S. Vidhyalakshmi, G. Sumathi, "Integral solutions of ternary bi-quadratic non-homogeneous equation $(\alpha + 1)(x^2 + y^2) + (2\alpha + 1)xy = z^4$ ", JARCE, Vol.6(2), (**July-December 2012**),Pp: 97-98.*

[8] *M.A. Gopalan, G. Sumathi, S. Vidhyalakshmi, "Integral solutions of ternary non-homogeneous bi-quadratic equation $(2k + 1)(x^2 + y^2 + xy) = z^4$ ", Indian Journal of Engineering, Vol.1(1), (**2012**), Pp: 37-39.*

[9] *M.A. Gopalan, G. Sumathi, S. Vidhyalakshmi, "On the ternary bi-quadratic non-homogeneous equation $x^2 + ny^3 = z^4$ ", Cayley J.Math, Vol.2(2), (**2013**), Pp: 169-174.*

[10] *A. Vijayasankar, M.A. Gopalan, V. Kiruthika, "On the bi-quadratic Diophantine equation with three unknowns* $7(x^2-y^2)+x+y=8z^4$ *", International Journal of Advanced Scientific and Technical Research, Issue 8, Volume 1, (**January-February 2018**), Pp:52-57.*

[11] *S. Vidhyalakshmi, M. A. Gopalan, S. A. Thangam and Ö. Özer, "On ternary biquadratic Diophantine equation* $11(x^2-y^2)+3(x+y)=10z^4$ *",Notes on Number Theory and Discrete Mathematics, Vol. 25, No. 3, (**2019**), Pp: 65–71.*

[12] *A.Vijayasankar, Sharadha Kumar, M.A.Gopalan, "A Search for Integer Solutions To Ternary Bi-Quadratic Equation* $(a+1)(x^2+y^2)-(2a+1)xy=[p^2+(4a+3)q^2]z^4$ *", EPRA (IJMR), 5(12), (**Dec 2019**), Pp: 26-32.*

[13] *M.A. Gopalan, B. Sivakami, "Integral solutions of Quartic equation with four unknowns* $x^3+y^3+z^3=3xyz+2(x+y)w^3$ *",Antartica J. Math., 10 (2), (**2013**), Pp: 151-159.*

[14] *M.A. Gopalan, S. Vidhyalakshmi, K. Lakshmi, "On the bi-quadratic equation with four unknowns* $x^2+xy+y^2=(z^2+zw+w^2)^2$ *" ,IJPAMS, 5 (1), (**2012**), Pp:73-77.*

[15] *M.A. Gopalan, S. Vidhyalakshmi, A. Kavitha, "Integral solutions to the bi-quadratic equation with four unknowns* $(x+y+z+w)^2=xyzw+1$ *", IOSR, Vol.7(4), (**2013**), Pp: 11-13.*

[16] *K. Meena, S. Vidhyalakshmi, M.A. Gopalan, S. Aarthy Thangam, "On the bi-quadratic equation with four unknowns* $x^3+y^3=39zw^3$ *", International Journal of Engineering Research Online, Vol. 2(1), (**2014**), Pp:57-60.*

[17] *M.A. Gopalan, V. Sangeetha, Manju Somanath, "Integer solutions of non-homogeneous biquadratic equation with four unknowns* $4(x^3-y^3)=31(k^2+3s^2)zw^2$ *" ,Jamal Academic Research Journal, Special Issue, (**2015**), Pp:296-299.*