Smooth Labeling and Semi Smooth Graceful Labeling operations on Various Graphs

Dr. Om Teraiya
Senior Assistant Professor & Head,
Atmiya University, RAJKOT − 360005.
E−mail: *om.teraiya@gmail.com*

Abstract

In this paper, I focus on α- labeling, Smooth graceful labeling of step grid graph. I also prove some result of semi smooth graceful Labeling on various graphs. I also prove the relation between semi smooth graceful graph and α- graceful graph. I also introduce the concept of Universal Graph and Non Universal Graph and derive the result.

Key words : α- labeling, Smooth graceful labeling, semi smooth graceful graph, universal graph, non universal graph.

AMS **subject classification** (2010) **:** 05C78.

1 Introduction

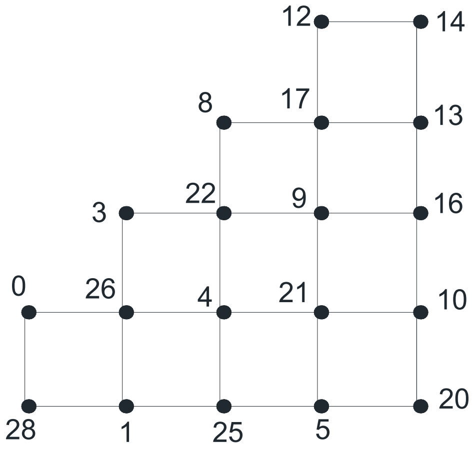
In 1966 Rosa [3] defined α-labeling as a graceful labeling with an additional property that there is an integer k $(0 \leq k < |E(G)|)$ such that for every $e = (x, y)$ $\in E(G)$, either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. A graph which admits α-labeling is necessarily bipartite graph with partition of $V(G) = V_1 \bigcup V_2$, where $V_1 = \{v \in V(G)/f(x) \leq k\}$ and $V_2 = \{v \in V(G)/f(x) > k\}$. We call such graph G with α-labeling f as an α-graceful graph.

In [5] Kaneria, Viradia and Makadia proved that the path union of a semi smooth graceful graph, star of a semi smooth graceful graph and cycle graph of a semi smooth graceful graph are graceful. They also proved step grid graphs St_n, Cycle of graphs $C(t \cdot H)$ and $C^m(t \cdot C_n)$ are smooth graceful graphs, where $t \equiv 0$ (mod 4), $n \equiv 0$ (mod 4), $m \in N$ and H is a semi smooth graceful graph. Some of these results we discuss here and we also discuss equivalentness of α-labeling and semi smooth graceful labeling.

2 Graceful Labeling and α-labeling

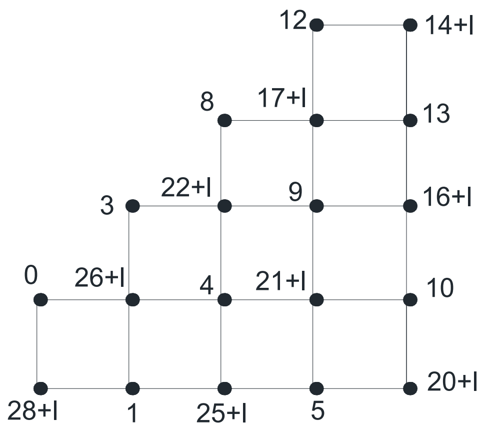
A bipartite graceful graph G with graceful labeling f is said to be *smooth graceful graph* if it admits an injective function $g : V(G) \longrightarrow \{0, 1, \dots, \lfloor \frac{q-1}{2} \rfloor, \lfloor \frac{q+1}{2} \rfloor + l, \lfloor \frac{q+2}{2} \rfloor + l, \dots, q + l\}$ such that its induced edge labeling map $g^\star : E(G) \longrightarrow \{1 + l, 2 + l, \dots, q + l\}$ defined as $g^\star(e) = |g(x) - g(y)|$, for every edge $e = (x, y) \in E(G)$, where l be any positive integer. A *semi smooth graceful graph* G, we mean it is a bipartite graph with $|E(G)| = q$ and the property that for all $l \in N$, there is an integer t $(1 \leq t \leq q)$ and an injective function $g : V(G) \longrightarrow \{0, 1, \dots, t-1, t+l, t+l+1, \dots, q+l\}$ such that the induced edge labeling function $g^\star : E(G) \longrightarrow \{1 + l, 2 + l, \dots, q + l\}$ defined as $g^\star(e) = |g(u) - g(v)|$ is a bijection for every edge $e = (u, v) \in E(G)$.

1.2.1 Illustration :α-labeling and smooth graceful labeling for St_5 Path union of graph :



figure−1.1

α-labeling for St_5



figure−1.1a

Smooth graceful labeling for St_5.

1.2.2 Definitions of path union, double path union

Let G be a graph and $G^{(1)}, G^{(2)}, \ldots, G^{(n)}$, $n \geq 2$ be n copies of G. Let $v \in V(G)$. The graph obtained by joining vertex v of $G^{(i)}$ with same vertex of $G^{(i+1)}$ by an edge, $\forall\ i = 1, 2, \ldots, n-1$ is called *path union of graph* G. We shall denote it by $P(n \cdot G)$. Here $P(n \cdot G)$ can obtain $|V(G)|$ different ways. If $G = k_1$ then $P(n \cdot K_1) = P_n$.

Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $G^{(0)}, G^{(1)}, \ldots, G^{(n)}$ be $n+1$ copies of G. Now each vertex v_i of $G^{(0)}$ join with the corresponding vertex v_i of $G^{(i)}$, $\forall\ i = 1, 2, \ldots, n$. Such graph is known as *star of* G and it is denoted by $G^{\star}$. We call $G^{(0)}$ as central copy of $G^{\star}$. $K_1^{\star} = K_2$ and $K_2^{\star} = P_6$.

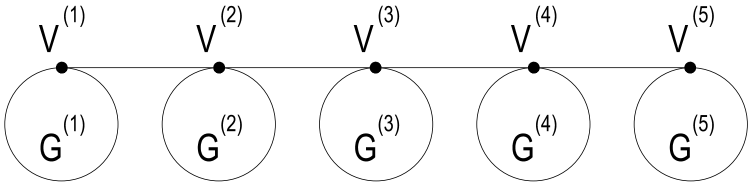
Let G be a graph and $G^{(1)}, G^{(2)}, \ldots, G^{(n)}$, $n \geq 2$ be n copies of G. Let $v \in V(G)$. The graph obtained by joining vertex v of $G^{(i)}$ with same vertex of $G^{(i+1)}$ by an edge, $\forall\ i = 1, 2, \ldots, n-1$ and v of $G^{(n)}$ with the same vertex of $G^{(1)}$ by an edge is called *cycle of* G. It is denoted by $C(n \cdot G)$. If we replace G by $C(n \cdot G)$ then such graph $C(n \cdot C(n \cdot G))$, we denote it by $C^2(n \cdot G)$. In general for any $t \geq 2$, $C^t(n \cdot G) = C(n \cdot C^{t-1}(n \cdot G))$. obviously $C(n \cdot K_1) = C_n$.

The *cartesian product of graphs* G and H is denoted as $G \times H$, is the graph with vertex set $V(G) \times V(H) = \{(u, v) : u \in V(G) \text{ and } v \in V(H)\}$ and vertex (u, v) adjacent to another vertex (x, y) if and only if either $u = x$ and $(v, y) \in E(H)$ or $v = y$ and $(u, x) \in E(G)$. $(P_n \times P_m)$ is known as *grid graph* on mn vertices.

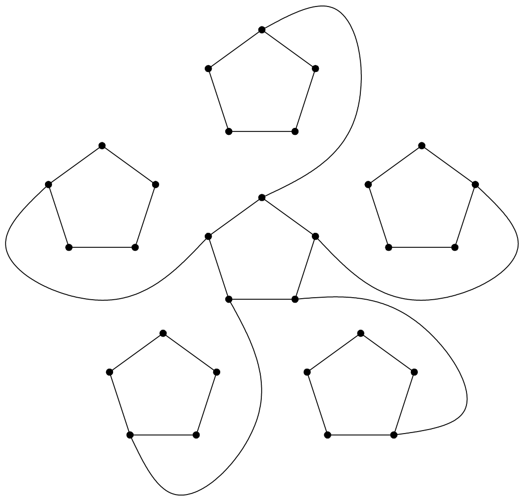
Let G be a graph and $G^{(1)}, G^{(2)}, \ldots, G^{(n)}$ $(n \geq 2)$ be n copies of G. Then the graph obtained by joining a pair of distinct vertices u, v of $G^{(i)}$ with same vertices of the graph $G^{(i+1)}$ by two edges, $\forall\ i = 1, 2, \ldots, n-1$ is called *double path union* of n copies of the graph G, such graph we can obtain $\frac{p(p-1)}{2}$ different ways, where $p = |V(G)|$ and we shall denote such grah by $D(n \cdot G)$. $D(n \cdot K_1)$ is undefined and $D(n \cdot P_2) = P_2 \times P_n$.

Take $P_n, P_n, P_{n-1}, P_{n-2}, \ldots, P_3, P_2$ paths on $n, n, n-1, n-2, \ldots, 3, 2$ vertices and arranged them vertically. A graph obtained by joining horizontal vertices of given successive paths is known as a *step grid graph of size* n $(n \geq 3)$ and we will denote it by St_n. Obviously $|V(St_n)| = \frac{1}{2}(n^2 + 3n - 2)$ and $|E(St_n)| = n^2 + n - 2$.

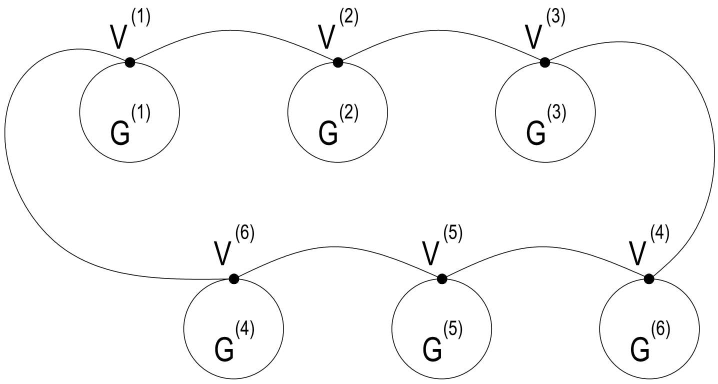
1.2.3 Illustration of above definitions :



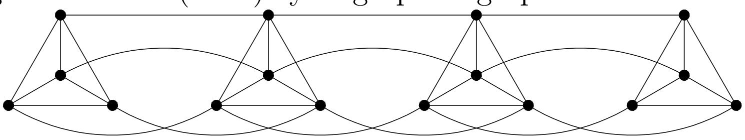
figure−1.2 $P(5 \cdot G)$ at a vertex $v \in V(G)$



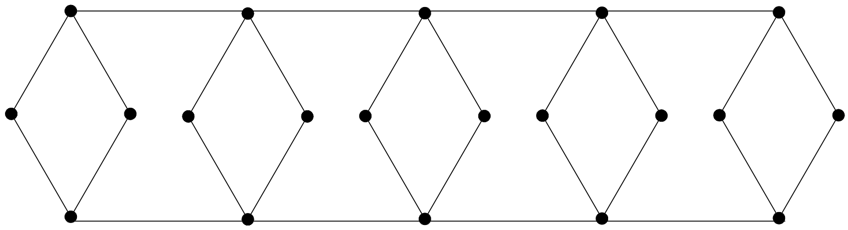
figure−1.3 $C_5^*(Star of C_5)$



figure−1.4 $C(6 \cdot G)$ cycle graph of graph G at a vertex v



figure−1.5 $P_4 \times W_3$



figure−1.6 $D(5 \cdot C_4)$ Double path union of C_4 at vertex u and v , where $d(u,v) = 2$ in C_4

In [4] Kaneria and Meera proved that $B_{m,n}$ is a semi smooth graceful graph. By taking $l = 0$ we get α-graceful labeling for $B_{m,n}$.

1.2.4 Illustration : Semi smooth graceful labeling for $B_{4,5}$ and its α-graceful labeling.

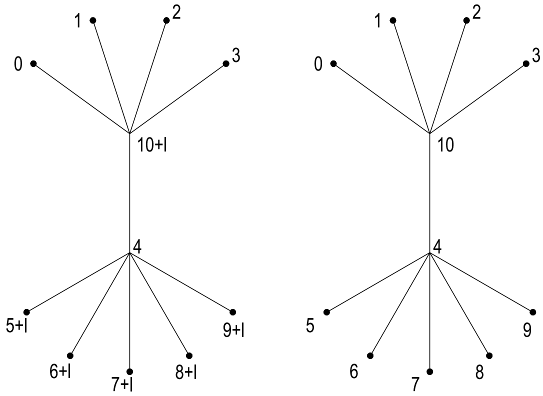
Here $V(B_{4,5}) = \{v_0, v_1, v_2, v_3, v_4, u_0, u_1, u_2, u_3, u_4, u_5\}$

and $E(B_{4,5}) = \{v_0v_1, v_0v_2, v_0v_3, v_0v_4, v_0u_0, u_0u_1, u_0u_2, u_0u_3, u_0u_5\}$. In [?] they have defined vertex labeling function $f : V(B_{4,5}) \longrightarrow \{0, 1, 2, 3, 4, , 5 + l, \ldots 10 + l\}$ as follows.

$f(v_0) = 10 + l,$ $\qquad f(u_0) = 4,$

$f(v_i) = i - 1,$ $\qquad \forall\ i = 1, 2, \ldots, 4;$

$f(u_j) = 10 + l - j,$ $\qquad \forall\ j = 1, 2, \ldots, 5.$



figure−1.7 $B_{4,5}$ and it's semi smooth and α-graceful labelings

1.2.5 Theorem : $B_{m,n}$ is α-graceful graph.

Proof : It is obvious that $V(B_{m,n}) = \{v_0, v_1, \dots v_m, u_0, u_1, \dots u_n\}$ and $E(B_{m,n}) = \{v_0v_1, v_0v_2, v_0v_3, \dots v_0v_m, v_0u_0, u_0u_1, \dots, u_0u_n\}$. Define $f : V(B_{m,n}) \longrightarrow \{0, 1, 2, 3, \dots m+n+1\}$ as follows.

$$f(v_i) = \begin{cases} q & if \quad i = 0 \\ i-1 & if \quad i = 1, 2, ..., m \end{cases}$$

Note that $f(V(B_{m,n})) = \{q, 0, 1, \dots, m-1, m+1, \dots m+n\}$ where $q = m+n+1$. So f is a bijective map and so, it is injective. Further we see that

$f^*(u_0v_0) = n+1$,

$f^*(v_0v_i) = q+1-i, \forall i = 1, 2, \dots m$ ·

$f^*(u_0v_j) = j, \forall j = 1, 2, \dots n$.

Hence $f^*(E(B_{m_1})) = \{n+1, q, q-1, \dots, n+2, 1, 2, \dots, n\} = \{1, 2, \dots, m+n+1\}$. and so $f* :\ E(B_{m,n}) \longrightarrow \{1, 2, \dots, m+n+1\}$ is a bijective map. So, f is a graceful labeling for $B_{m,n}$.

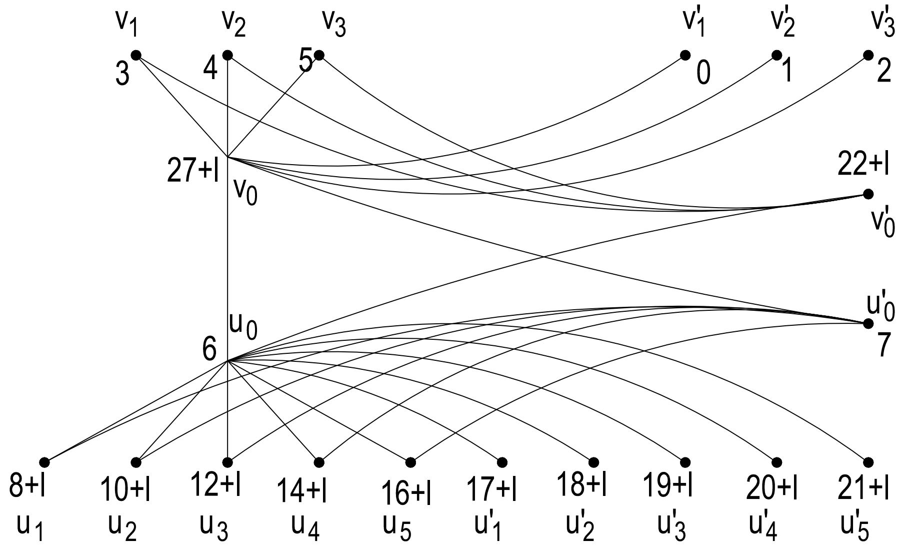
Take $k = m$. Observe that mini $\{f(u_0), f(v_0)\}$ = m = k $\leq$ k $<$ max$\{f(u_0), f(v_0)\}$ = q, mini $\{f(v_0), f(v_i)\}$ = i-1 $\leq$ m-1 $\leq$ k $<$ max$\{f(v_0), f(v_i)\}$, $\forall i = 1, 2, 3 \dots, m$. and mini $\{f(u_0), f(v_j)\}$ = m = k $\leq$ k $<$ max$\{f(u_0), f(u_j)\} = m + j$, $\forall j = 1, 2, 3 \dots, n$. Therefore, each $xy \in E(B_{m,n})$ mini $\{f(x), f(y)\} \leq$ k $<$ max$\{f(x), f(y)\}$. Hence, f is an α-graceful labeling for $B_{m,n}$ and so, it is an α-graceful graph.

In [5] Kaneria, Makadia and Viradia proved that splitting graph of $B_{m,n}$ is a semi smooth graceful graph.

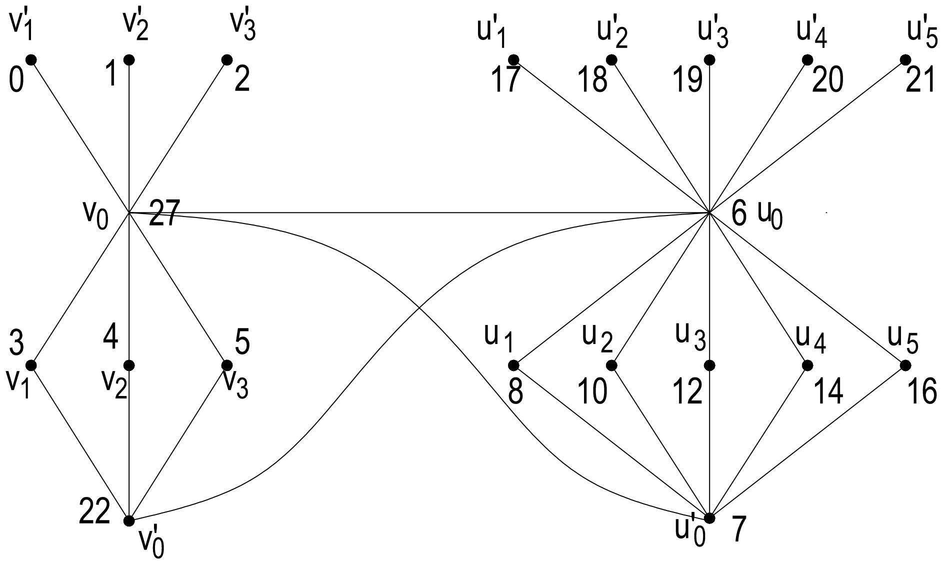
1.2.6 Illustration : Semi smooth graceful labeling and its α-graceful labeling for $B_{3,5}$.

Here $V(S^{'}(B_{3,5})) = \{v_0, v_1, v_2, v_3, u_0, u_1, \dots, u_5, v^{'}_o, v^{'}_1, v^{'}_2, v^{'}_3, u^{'}_o, u^{'}_1, \dots u^{'}_5\}$ and $E(S^{'}(B_{3,5})) = \{v_0v_1, v_0v_2, v_0v_3, v_0u_0, u_0u_1, u_0u_2, \dots u_0u_5, v_0v^{'}_1, v_0v^{'}_2, v_0v^{'}_3, v^{'}_0v_1, v^{'}_0v_2, v^{'}_0v_3, v^{'}_0u_0, u^{'}_0v_0, u^{'}_0u_1, u^{'}_0u_2, \dots, u^{'}_0u_5, u_0u^{'}_1, \dots, u_0u^{'}_5\}$. In [3] Kaneria, Makadia and Viradia have defined

$f(V(S^{'}(B_{3,5}))) \longrightarrow \{0, 1, 2, \ldots, 7+l, 8+l, \ldots, 27+l\}$ as follows.
$f(v_0) = 27+l, f(u_0) = 6, f(v_0^{'}) = 22+l, f(u_0^{'}) = 7;$ $f(v_i) = 2+i, \forall i = 1,2,3;$ $f(v_2^{'}) = 3-i, i = 1,2,3;$ $f(u_j) = 6+2j+l, \forall j = 1,2,3,4,5$ and $f(u_j^{'}) = 16+j+l,$ $\forall j = 1,2,3,4,5.$



figure−1.8 Semi smooth graceful labeling of $S^{'}(B_{3,5})$



figure−1.9 α-graceful labeling of $S^{'}(B_{3,5})$

1.2.7 Theorem : $S^{'}(B_{m,n})$ is a α-graceful graph.

Proof : Let $G = S^{'}(B_{m,n})$, $V(G) = \{v_0, v_1, v_2, \ldots, v_m, u_0, u_1, \ldots, u_n, u_o^{'}, u_1^{'}, \ldots, u_n^{'}, v_0^{'}, v_1^{'}, v_2^{'}, \ldots, v_m^{'}\}$ and $E(G) = E(B_(m,n)) \bigcup \{v_0v_1^{'}, v_0v_2^{'}, \ldots, v_0v_m^{'}, v_0^{'}v_1, v_0^{'}v_2, \ldots, v_0^{'}v_m, u_0u_1^{'}, u_0u_2^{'}, \ldots, u_0u_n^{'}, u_0^{'}u_1, u_0^{'}u_2, \ldots, u_0^{'}u_n, u_0^{'}v_0, u_0v_0^{'}\}$.
$i.e.$ $|V(G)| = 2(m+n+1)$ and $|E(G)| = 3(m+n+1)$. Define a vertex labeling function $f : V(G) \longrightarrow \{0, 1, 2, \ldots, 3(m+n+1)\}$ as follows.

$f(v_0) = 3(m+m+1)$, $f(u_0) = 2m$, $f(v_0^{'}) = 2m+3n+1$, $f(u_0^{'}) = 2m+1$, $f(v_i) = m+(i-1), \forall i = 1, 2, 3, \ldots, m$; $f(v_i^{'}) = m-i, \forall i = 1, 2, \ldots, m$; $f(u_j) = 2(m+j)$; $\forall j = 1, 2, 3, \ldots, n$ and $f(u_j^{'}) = 2(m+n)+j, \forall j = 1, 2, \ldots, n$.

Note that $f(V(G)) = \{0, 1, \ldots, m-1, m, m+1, \ldots, 2m-1, 2m, 2m+1, 2m+2, 2m+4, \ldots, 2m+2n, 2m+2n+1, \ldots, 2m+3n, 2m+3n+1, 3(m+n+1)\}$ $\subset \{0, 1, \ldots, 3(m+n+1)\}$ and f is an injective map, as there is no $x, y \in V(G)$ such that $f(x) = f(y)$. Let $q = 3(m+n+1)$. Observe that

$f^{*}(v_0v_i^{'}) = q - m + i, \forall i = 1, 2, ..., m;$

$f^{*}(v_0v_i) = q - m - i + 1, \forall i = 1, 2, ..., m;$

$f^{*}(v_0u_0) = m + 3n + 3;$

$f^{*}(v_0u_0^{'}) = m + 3n + 2;$

$f^{*}(v_0^{'}v_i) = m + 3n + 2 - i, \forall i = 1, 2, ..., m;$

$f^{*}(v_0^{'}u_0) = 3n + 1;$

$f^{*}(u_0u_j^{'}) = 2n + j, \forall j = 1, 2, ...n$

$f^{*}(u_0u_j) = 2j, \forall j = 1, 2, ...n;$

$f^{*}(u_0^{'}u_j) = 2j - 1, \forall j = 1, 2, ...n;$

Hence, $f^{*}(E(G)) = \{1, 2, \ldots, q\}$ and f^{*} is an injective map as there is no $e_1, e_2 \in E(G)$ such that $f^{*}(e_1) = f^{*}(e_2)$. Therefore, f^{*} is a bijective map. So, f is a graceful labeling for G. Take $k = F^{*}(u_0^{'} = 2m+1$. Further we see that

$\min\{f(v_0), f(v_i^{'})\} < m < k < \max\{f(v_0), f(v_i^{'})\} = q,$

$\min\{f(v_0), f(v_i)\} < 2m < k < \max\{f(v_0), f(v_i)\} = q,$

$\min\{f(v_0), f(u_0)\} = 2m < k < \max\{f(v_0), f(u_0)\} = q,$

$$\min\{f(v_0), f(u_0^{'})\} = 2m+1 \leq k < \max\{f(v_0), f(u_0^{'})\} = q,$$
$$\min\{f(v_0^{'}), f(v_i)\} = 2m < k < \max\{f(v_0^{'}), f(v_i)\} = 2m+3n+1,$$
$$\min\{f(v_0^{'}), f(u_0)\} = 2m < k < \max\{f(v_0^{'}), f(u_0)\} = 2m+3n+1,$$
$$\min\{f(u_0), f(u_j^{'})\} = 2m < k < 2m+2n+1 \leq \max\{f(u_0), f(u_j^{'})\},$$
$$\min\{f(u_0), f(u_j)\} = 2m < k < 2m+2 \leq \max\{f(u_0), f(u_j)\},$$
$$\min\{f(u_0^{'}), f(u_j)\} = 2m+1 \leq k < 2m+2 \leq \max\{f(u_0^{'}), f(u_j)\},$$
$$\forall i = 1, 2, \ldots, m \text{ and } \forall j = 1, 2, \ldots, n.$$

Therefore, each $xy \in E(G)$, $min\{f(x), f(y)\} \leq k < max\{f(x), f(y)\}$ and so, f is an α-graceful labeling for G. Hence,$G = S^{'}(B_{m,n})$ is a α-graceful graph.

3 Semi Smooth Graceful Labeling and α-labeling

In [5] Kaneria, Makadia and Viradia Proved that every smooth graceful graph G is also an α-graceful graph and every α-graceful graph G admits a semi smooth graceful labeling. In fact by definitions of smooth graceful and semi smooth graceful labelings, every smooth graceful graph is a semi smooth graceful graph, as by taking $\lfloor \frac{q+1}{2} \rfloor = t$, smooth gracefulness is a particular case of semi smooth gracefulness. In this section we shall prove that every semi smooth graceful graph G admits an α-graceful labeling. We also discuss about universal graceful labeling and universal α-graceful labeling.

1.3.1 Theorem : Every semi smooth graceful graph G is also an α-graceful graph.

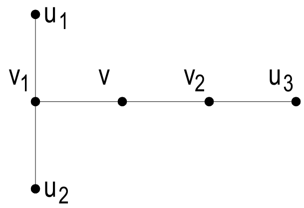
Proof : Let G be a semi smooth graceful graph with a semi smooth graceful labeling. $g : V(G) \longrightarrow \{0, 1, \ldots, t-1, t+l, t+l+1, \ldots, q+l\}$ for some $t \in N$ and for every $l \in N$, its edge induced labeling function $g^* : E(G) \longrightarrow \{1+l, 2+l, \ldots, q+l\}$ defined by $g^*(xy) = |g(x) + g(y)|$ is a bijective map.

By taking $l = 0$, g becomes a graceful labeling for G and so, G is graceful labeling, i.e. g is an injective map and $g^* : E(G) \longrightarrow \{1, 2, \ldots, q\}$ is bijective map.

Take $k = t - 1$. It can be observe that G is a bipartite graph with $V(G)$ has a partition $v_1 = \{v \in V(G)/g(v) \leq t - 1\}$ and $v_2 = \{v \in V(G)/g(v) \geq t\}$. i.e. $v_2 = \{v \in V(G)/g(v) > t\}$. Thus for any $xy \in E(G)$ we get $\min\{g(x), g(y)\} \leq k = t - 1 < \max\ \{g(x), g(y)\}$ and so, g is an α-graceful labeling g by taking $l = 0$ and so, G is an α-graceful graph.

Let G be a graceful graph with graceful labeling $f : V(G) \longrightarrow \{0, 1, \ldots, E(G)\}$. A vertex $v \in V(G)$ is called a graceful center w.r.t f if $f(v) = 0$. A graph G is said to be *universal graceful graph* if for any $v \in V(G)$, v is a graceful center for G w.r.t. some graceful labeling g of G. Also we call G is an *universal α-graceful graph* if for every $v \in V(G)$ there exits an α-graceful labeling g on G and $g(v) = 0$. As $q - f$ is also a graceful labeling for G $(g - f)(w) = 0$,where $w \in V(G)$ and $f(w) = q$, w is also a graceful center w.r.t $q - f$ for G, where $q = |E(G)|$. Thus, any graceful graph G has at least two graceful center v and w, when $f(v) = 0$ and $f(w) = 0$.

In [2] Makadia, Karvadiya and Kaneria proved that any α-graceful graph G has at least four α-graceful center, as f, $q - f$, h, $q - h$ four α-labelings for G, where $h : V(G) \longrightarrow \{0, 1, \ldots, q\}$ defined by $h|v_1 = k - f$ and $h|v_2 = q + k + 1 - f$, where $k = min\{f(x), f(y)\}$ and $xy \in E(G)$ such that $f^*(xy) = 1$, $v_1 = \{v \in V(G)/f(v) \leq k\}$ and $V_2 = V(G) - V_1$. In some paper they have proved that the following tree T is an α-graceful graph, but it is not a universal graph, as the vertex v can not be a graceful center for T w.r.t any graceful labeling T.



figure−1.10 Non universal graceful graph

Every cycle $C_n(n \equiv 0 \pmod 4)$, star $K_{1,n}$ are universal α-graceful graphs. While $C_n(n \equiv 3 \pmod 4)$ and W_n are universal graceful graphs but they are not universal α-graceful graphs.

1.3.2 Theorem : The graph obtained by joining two copies of a universal α-graceful graph G say $G^{(1)}$ and $G^{(2)}$ by an edge with any two corresponding vertices $v^{(1)} \in V(G^{(1)})$ and $v^{(2)} \in V(G^{(2)})$, for some $v \in V(G)$ is also a universal α-graceful graph.

Proof : Let H is a graph obtained by two copies of a universal α-graceful graph G say $G^{(1)}$ and $G^{(2)}$ by an edge with any two corresponding vertices $v^{(1)} \in V(G^{(1)})$ and $v^{(2)} \in V(G^{(2)})$, for some $v \in V(G)$.

Let $u \in V(H)$. Implies either $u \in V(G^{(1)})$ or $u \in V(G^{(2)})$. We may assume that $u \in V(G^{(1)})$, call u by u^1, as $u^1 \in V(G^{(1)})$. Now $u \in V(G)$ and since, G is a universal α-graceful graph, there exit an α-labeling $f : V(G) \longrightarrow \{0, 1, \ldots, |E(G)|\}$ such that $f(u) = 0$, since f is an α-graceful labeling for G, there exit k $(0 \leq k < |E(G)|)$ such that for each $xy \in E(G)$, $\min\{f(x), f(y)\} \leq k < max\{f(x), f(y)\}$. Take $v_1 = \{w \in V(G)/f(w) \geq k\}$ and $V_2 = V(G) - V_1$.

Here H is the graph obtained by joining $G^{(1)}$ and $G^{(2)}$, two copies of G by an edge between $v^{(1)}$ and $v^{(2)}$, where $v^{(1)} \in V(G^{(1)})$ and $v^{(2)} \in V(G^{(2)})$ for some $v \in V(G)$. It is obvious that $V(H) = V(G^{(1)}) \bigcup V(G^{(2)})$ and $E(H) = E(G^{(1)}) \bigcup E(G^{(2)}) \bigcup \{(v^{(1)}), (v^{(2)})\}$, $|V(H)| = 2|V(G)|$ and $|E(H) = 2q + 1$, where $q = |E(G)|$.

Define $g : V(H) \longrightarrow \{0, 1 \ldots, 2q+1\}$ as follows. $g|V_1^{(1)} = f|V^{(1)}$, $g|V_2^{(2)} = f|V^{(2)}$, $g|V_1^{(2)} = f|V^{(2)} + q + 1$ and $g|V_2^{(1)} = f|V^{(1)} + (q+1)$. Since f is $1-1$, g is also a $1-1$ map.Take $(w_1 w_2) \in E(G)$ be any edge for any $i = 1, 2$.

$g^*(w_1^i, (w_2^i) = |g(w_1^i) - g(w_2^i)|$.

$$= \begin{cases} (q+1) + f(w_1) - f(w_2), & if \quad w_1^{(i)} \in V_2^{(1)} \cup V_1^{(2)} \\ (q+1) + f(w_1) - f(w_2), & if \quad w_1^{(i)} \in V_2^{(2)} \cup V_1^{(1)} \end{cases}$$

$$= \begin{cases} (q+1) + f^*((w_1, w_2)), & if \qquad w_1 \in V_1 \quad \& \quad i = 2 \quad or \\ (q+1) - f^*((w_1, w_2)), & otherwise. \end{cases}$$

Since range of $f^* = \{1, 2, 3, \ldots, q\}$, $g^*(v^{(1)}, v^{(2)}) = q + 1$, range of $g^* = \{1, 2, \ldots, 2q+1\}$. Thus $g^* : E(H) \longrightarrow \{1, 2, \ldots, 2q+1\}$ is a bijective map, as g^* is $1-1$ and range of $g^* =$ codomain of g^*. Hence g^* is an graceful labeling for H. Take $k = q$. Now for each $u \in V(G)$, $f(u) \leq q$. Therefore, $\min\{g(u^{(1)}), g(u^{(2)})\} \leq q$ and $\max\{g(u^{(1)}), g(u^{(2)})\} \geq q + 1$.

Therefore, $\min\{g(u^{(1)}), g(u^{(2)})\} \leq k < max\{g(v^{(1)}), g(v^{(2)})\}$. For any $(w_1^{(i)}, w_2^{(i}) \in E(H)$, we have $(w_1, w_2) \in E(G)$ and so one of them lies in V_1 and another of them lies in V_2, as G is a bipartite graph.

For any $(w_1^{(i)}, w_2^{(i)}) \in E(H)$, min $\{g(w_1^{(i)}), g(w_2^{(i)})\} \leq k < max\{g(w_1^{(i)}), g(w_2^{(i)})\}$ for any $i = 1, 2$. Hence for any $(w_1, w_2) \in E(H)$ and min $\{g(w_1, g(w_2)\} \leq k < max\{g(w_1, g(w_2)\}$. Therefore, g is an α-graceful labeling for H.

Moreover $g(u^{(1)}) = f(u) = 0$, as $u \in V_1$. So u is also an α-graceful center for H w.r.t. α-graceful labeling g of H. Since $u \in V(H)$ be the arbitrary vertex of H, H must be universal α-graceful graph.

4 Conclusive Remarks

In this paper I have discussed graceful labeling, α-labeling, smooth graceful graphs, Semi smooth graceful graphs and applied α-labeling on some known graphs in detail. The discussion includes definitions and known results for labeling. We proved some new results based on α-labeling on some known graphs.

5 References

1. J. A. Gallian, A Dynamic Survey of Graph Labeling, *The Electronics J. of Combinatorics*, 21(2018), $\#D56$.

2. H Karvadiya, H M Makadia and V J Kaneria, Graceful Centre of graceful graphs and Alpha graceful graph.

3. A. Rosa On certain valuation of graph. *Theory of Graphs, Rome, Goden and Breach (N.Y. Paris)*, 1967 : 349355, 1966.

4. Meera Meghpara, V J Kaneria. Semi smooth graceful labeling on some graphs.*International Journal of Mathematics And its Applications*, 3 : 15, 2015.

5. R V Viradia, V J Kaneria and H M Makadia. Various graph operations on semi smooth graceful graphs. *Int. J. Math. and Soft Computing*, 6(1) : 5779,2016.