A Subclass of Harmonic Univalent functions

Dr.Noohi Khan(AP II)

Amity School of Applied Sciences

Amity University Lucknow

Abstract

In this paper we introduce and study a class $HP_{\lambda}(\beta)$, $(0 \le \beta < 1, \quad 0 \le \lambda < 1)$ consisting of harmonic and univalent functions $f = h + \bar{g}$ for which $\operatorname{Re}\{h_{\lambda}^{'}(z) + g_{\lambda}^{'}(z)\} > \beta$ where h and g are analytic in U. We give sufficient coefficient condition for normalized harmonic functions in the class $HP_{\lambda}(\beta)$. This condition is also necessary for negative coefficients.

Key words: Harmonic functions, coefficient inequality, extreme points, Harmonic Univalent functions

1. INTRODUCTION

1. Let U denotes the open unit disk and S_H denotes the class of all complex valued , harmonic sense preserving univalent functions f in U normalized by

$$f(0)=0, \qquad f_z(0)=1$$

each $f \in S_H$ can be expressed as

(1.1) $f = h + \bar{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

And both h and g are analytic in U.

A necessary and sufficient condition for f to be locally univalent and sense preserving in U is

$$|h'(z)| > |g'(z)| \quad \text{in U}$$

Where h and g are given by (1.1).

A function of the form (1.1) is said to be harmonic starlike for $|z|=r < 1$ if

$$\frac{\partial}{\partial\theta}(\arg f(re^{i\vartheta}) = \operatorname{Re}\left\{\frac{zh'(z)+\overline{zg'(z)}}{h(z)+\overline{g(z)}}\right\} > 0, \quad |z| = r < 1$$

See in [1].

Silverman [2] proved that the coefficient conditions

$$\sum_{n=2}^{\infty} n(|a_n|+|b_n|) \leq 1 \quad \text{and}$$

$$\sum_{n=2}^{\infty} n^2(|a_n|+|b_n|) \leq 1$$

are necessary and sufficient for functions f = h+g to be harmonic starlike and harmonic convex respectively.

Karpuzogullari et. al [3] defined and studied a class HP(β) and its subclass HP*(β) with negative coefficients of S_H with the condition $\operatorname{Re}\{h'(z)+g'(z)\} > \beta$. Motivated with the work of Karpuzogullari [3] we define in this paper two sub classes of S_H namely $HP_\lambda(\beta)$ and $HP_\lambda^*(\beta)$ as follows:

(1.2) Definition of $HP_\lambda(\beta)$ and $HP_\lambda^*(\beta)$

A function of the form (1.1) belonging to S_H is said to be in $HP_\lambda(\beta)$ class if and only if

(1.3) $\operatorname{Re}\{h_{\lambda'}(z)+g_{\lambda'}(z)\} > \beta, 0 \leq \beta \leq 1$

Where,

$$h_\lambda(z) = (1-\lambda)h + \lambda zh'$$
$$g_\lambda(z) = (1-\lambda)g + \lambda zg' \quad , \quad 0 \leq \lambda \leq 1$$

We further denote by $HP_\lambda^*(\beta)$ the sub class of $HP_\lambda(\beta)$ whose members

(1.4) $f = h + \overline{g}$

With h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \qquad \text{and}$$

$$g(z) = -\sum_{n=1}^{\infty} |b_n| z^n$$

It is noted that $HP_o(\beta) \equiv HP(\beta)$

$HP_o*(\beta) \equiv HP*(\beta)$

In particular if $o \le \beta_1 \le \beta 2 < 1,$ then

$$HP_\lambda(\beta_2) \subset HP_\lambda(\beta_1)$$

2.Main Results

Theorem 2.1 Let $f = h + \overline{g}$ of the form (1.1) and

(2.1)

$$\sum_{n=1}^{\infty} n(1 - \lambda + n\lambda)\left(|a_n| + |b_n|\right) \le 2 - \beta; \qquad 0 \le \lambda < 1$$

Where $a_1 = 1$ and $0 \le \beta < 1$ then, f is harmonic univalent, sense preserving in U and $f \varepsilon HP_\lambda(\beta).$

The result is sharp.

Proof - Let $|z_1| \le |z_2| < 1$ and $f = h + \overline{g}$ of the form (1.1) we have,

$$|f(z_1) - f(z_2)| \ge |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)|$$

$$\ge |z_1 - z_2| \left[1 - \sum_{n=2}^{\infty} n|a_n||z_2|^{n-1} - \sum_{n=1}^{\infty} n|b_n||z_2|^{n-1}\right]$$

$$\ge |z_1 - z_2| \left[1 - \sum_{n=2}^{\infty} n|a_n| - \sum_{n=1}^{\infty} n|b_n|\right]$$

$$\ge |z_1 - z_2| \left[2 - \sum_{n=1}^{\infty} n(|a_n| + |b_n|)\right]$$

$$\geq |z_1 - z_2|\left[2 - \sum_{n=1}^{\infty}(1-\lambda+n\lambda)(|a_n|+|b_n|)\right]$$

$$\geq |z_1 - z_2|[2-(2-\beta)]$$ (Using (2.1))

>0

Hence, f is univalent in U.

fis also sense preserving in U. this is because

$$|h'(z)| \geq 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} > 1 - \sum_{n=2}^{\infty} n|a_n|$$

$$\geq 2 - \sum_{n=1}^{\infty} n|a_n| \geq 2 - \sum_{n=1}^{\infty} n(1-\lambda+n\lambda)|a_n|$$

$$\geq \beta + \sum_{n=1}^{\infty} n(1-\lambda+n\lambda)|b_n|$$ (Using (2.1))

$$\geq \sum_{n=1}^{\infty} n|b_n| > \sum_{n=1}^{\infty} n|b_n||z|^{n-1} > |g'(z)|$$

Now, we have to show that $f\varepsilon HP_{\lambda}(\beta)$ Using the fact that $R_e W \geq \beta$ if and only if $|1-\beta+w| \geq |1+\beta-w|$, it is sufficient to show that

(2.2) $|1-\beta+h_{\lambda'}(z)+g_{\lambda'}(z)| - |1+\beta-h_{\lambda'}(z)-g_{\lambda'}(z)| \geq 0$

Using the series expansion of $h_{\lambda'}(z)$ and $g_{\lambda'}(z)$ we get

$$\left|2-\beta+\sum_{n=2}^{\infty}n(1-\lambda+n\lambda)a_nz^{n-1}+\sum_{n=1}^{\infty}n(1-\lambda+n\lambda)b_nz^{n-1}\right|$$
$$-\left|\beta-\sum_{n=2}^{\infty}n(1-\lambda+n\lambda)a_nz^{n-1}-\sum_{n=1}^{\infty}n(1-\lambda+n\lambda)b_nz^{n-1}\right|$$

$$\geq 2\left[(1-\beta)-n(1-\lambda+n\lambda)\left\{|z|\sum_{n=2}^{\infty}n(1-\lambda+n\lambda)|a_n|+\sum_{n=1}^{\infty}n(1-\lambda+n\lambda)|b_n|\right\}\right]$$
≥ 0 (Using (2.1))

Hence, $f\varepsilon HP_{\lambda}(\beta)$

The harmonic mapping

(2.3) $$f(z)=z+\sum_{n=2}^{\infty}\frac{(1-\beta)x_nz^n}{n(1-\lambda+n\lambda)}+\sum_{n=1}^{\infty}\frac{(1-\beta)\overline{y_nz^n}}{n(1-\lambda+n\lambda)}$$

Where $\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1$

Shows that the coefficient bounds given by (2.1) are sharp. The functions of the form (2.3) are in $HP_{\lambda}(\beta)$ because

$$\sum_{n=1}^{\infty}n(1-\lambda+n\lambda)(|a_n|+|b_n|)=1+(1-\beta)\left(\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|\right)=1+1-\beta=2-\beta$$

Therefore, the result is sharp.

Corollary (2.2) The result of Karpuzogullari et-al [3] follows on putting $\lambda=0$ in theorem (2.1).

Theorem (2.3) Let be $f=h+\overline{g}$ be given by (1.4) then, $f\varepsilon HP_{\lambda}^{*}(\beta)$ if and only if

(2.4) $$\sum_{n=1}^{\infty}n(1-\lambda+n\lambda)(|a_n|+|b_n|)\leq 2-\beta$$

Where $a_1 = 1$ and $0\leq\beta<1$.

Proof - If part follows directly from theorem (2.1). Now for only if part we show that if $f \varepsilon HP_{\lambda}^{*}(\beta)$,

The condition (2.4) holds.

If $f = h + \overline{g}$ of the form (1.4) be in $HP_{\lambda}^{*}(\beta)$ for $0 \le \beta < 1$ we have

$$\operatorname{Re}\{h_{\lambda'}(z) + g_{\lambda'}(z)\} > \beta$$

Or equivalently

$$\operatorname{Re}\left\{1 - \sum_{n=2}^{\infty} n(1-\lambda+n\lambda)|a_n||z|^{n-1} - \sum_{n=1}^{\infty} n(1-\lambda+n\lambda)|b_n||z|^{n-1}\right\} > \beta$$

If we choose z to be real and let $z \to 1^{-}$, we get $1 - \sum_{n=2}^{\infty} n(1-\lambda+n\lambda)|a_n| - \sum_{n=1}^{\infty} n(1-\lambda+n\lambda)|b_n| \ge \beta$

Or

$$\sum_{n=2}^{\infty} n(1-\lambda+n\lambda)|a_n| + |b_n| \le 2 - \beta$$

Which is the required coefficient inequality (2.4) corollary (2.4) - let $f = h + \overline{g}$ be given by (1.4) then,

$f \in HP_{\lambda}^{*}(\beta)$ if and only if

(2.6) $$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \le \frac{1-\beta-|b_1|}{1+\lambda}$$

2. Conclusion

With the help of above coefficient inequality we obtain Distortion Bounds, extreme points convex combination, Radius of convexity and neighbourhoods for the class $HP_{\lambda}^{*}(\beta)$.

3. References

[1] J Clunie Sheil – Small; Harmonic Univalent functions, Ann. Acad Sci. Fenn Series A.I, Maths -9 (1984) 3-25.

[2] H-Silverman; Harmonic Univalent functions with negative coefficients J.Math Anal. Appl 220 (1998).

[3] Karpuzogullari; *MetinÖztuřk*, *Mü*min Yamankarancleniz, A Subclass of harmonic univalent functions with negative coefficients, applied Mathematics and computation 142 (2003) 469-476.

A Subclass of Harmonic Univalent functions

[4] St Ruschweyh, Neighbourhoods of Univalent functions, Proc. Amer. Maths Soc 81 (1981) 521-528.

[5] T. Sheil – Small, Constrants for planer harmonic mapping, J. London Math. Soc.2 (42) (1990) 237-248.