Generalized PAOR method for solving Non-square linear systems

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Abstract: In this paper, we present Parametric Accelerated Over Relaxation (PAOR) method for solving non-square linear systems and discuss about its convergence. A numerical example is considered to compare this method with the generalized SOR, generalized Jacobi and generalized Gauss-Seidel methods.

Keywords: Iterative method, Jacobi, Gauss-Seidel, SOR, AOR, Convergence

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1. Introduction:

The linear system of m equations in n unknowns (m<n) of the form

$$\begin{array}{c} x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{array} \tag{1.1}$$

can be put as a matrix system of the form

$$AX = b \tag{1.2}$$

where $A \in \mathbb{R}^{m\times n}$ and X and b are unknown and known n dimensional and m dimensional vectors respectively.
If the matrix A is partitioned into

$$A = [B\ \tilde{B}] \tag{1.3}$$

where $B \in \mathbb{R}^{m\times n}$ and $\tilde{B} \in \mathbb{R}^{m,n-m}$, then the equation (1.2) can be expressed in the form

$$[B\ \tilde{B}]\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = b \tag{1.4}$$

$where\ X_1\ and\ X_2$ are m and $n-m$ dimensional vectors respectively.

The system (1.2) can be written as

$$BX_1 + \tilde{B}X_2 = b \tag{1.5}$$

For $m = n\ in\ (1.1),$ the system (1.2) can be expressed as

$$(I - L - U)X = b \tag{1.6}$$

where $-L\ and\ -U$ are strictly lower and upper triangular parts of the coefficient matrix A respectively.

The Parametric Accelerated Over Relaxation (PAOR) method considered by V.B.Kumar,Vatti et.al [8] for solving (1.6) is defined by

$$[(1+\alpha)I-\omega L]X^{(k+1)}=\{(1+\alpha-r)I+(r-\omega)L+rU)\}X^{(k)}+rb \qquad (1.7)$$
$$(n=0,1,2\ldots)$$

where $\alpha\neq -1$.

The methods such as AOR, SOR, Guass-seidal and Jacobi cab be realized from (1.7) for the choices of $\alpha, r\ and\ \omega$ as

$$(\alpha,r,\omega)=(0,r,\omega),(0,\omega,\omega),(0,1,1),(0,1,0) \qquad (1.8)$$

The Iteration matrix of PAOR method is

$$P_{\alpha,r,\omega}=[(1+\alpha)I-\omega L]^{-1}\{\{(1+\alpha-r)I+(r-\omega)L+rU)\} \qquad (1.9)$$

and the choice of parameters as given by V.B.Kumar,Vatti et.al [8] are

(i) When $\underline{\mu}=\overline{\mu}\ and\ k=1$

$$\omega=\frac{2(1+\alpha)}{1+\sqrt{1-\overline{\mu}^2}}\ \&\ r=\frac{(1+\alpha)}{\sqrt{1-\overline{\mu}^2}} \qquad (1.10)$$

(ii) When $\underline{\mu}\neq\overline{\mu}\ and\ k>1$

$$\omega=\frac{2(1+\alpha)}{1+\sqrt{1-\overline{\mu}^2}}\ \&\ r=1+\alpha+\omega+\frac{\overline{\mu}^2-\underline{\mu}^2}{2} \qquad (1.11)$$

(ii) When $\underline{\mu}\neq\overline{\mu}\ and\ k<1$

$$\omega=\frac{2(1+\alpha)}{1+\sqrt{1-\overline{\mu}^2}}\ \&\ r=\left(1+\alpha+\omega+\frac{\overline{\mu}^2-\underline{\mu}^2}{2}\right)/2 \qquad (1.12)$$

where $\underline{\mu}$, $\overline{\mu}$ are the smallest and largest eigen values of the Jacobi matrix J in magnitude and

$$k=1-\sqrt{1-\overline{\mu}^2}+\frac{\frac{\omega\overline{\mu}^2}{2}}{\omega+\frac{\overline{\mu}^2-\underline{\mu}^2}{2}} \qquad (1.13)$$

2. Generalized Parametric Accelerated over relaxation (PAOR) method:

Let $y^{(k)}$ be the k^{th} approximate solution to the system(1.2). Then as mentioned in [1], the $(k+1)^{th}$ approximate is obtained as

$$y^{(k+1)}=y^{(k)}+S(A)d^{(k)} \qquad (2.1)$$

where $S(A)$ is the $n\times m$ matrix having $0,-1,1$ as its elements indicating the signs of the transpose matrix of A and

$$d^{(k)} = \left[\frac{b_1 - A_1 y^{(k)}}{m\|A_1\|}, \frac{b_2 - A_2 y^{(k)}}{m\|A_2\|}, \dots \dots \frac{b_m - A_m y^{(k)}}{m\|A_m\|}\right]^T \quad (2.2)$$

Here $\|A_i\|$ $(i = 1,2,,,,m)$ are l_1-norms of i^{th} row vector of A.

For the solution of (1.4) with an initial approximation

$X^{(0)} = \begin{pmatrix} X_1{}^{(0)} \\ X_2{}^{(0)} \end{pmatrix}$ one has to perform the following steps:

Step 1: Assign $k \leftarrow 0$

Step 2: Calculate for a positive definite matrix B, $\tilde{b}^{(k)} = b - BX_1{}^{(k)}$ (2.3)

Applying the procedure given in [1] to the system $\tilde{B}Y = \tilde{b}^{(k)}$ with the initial guess $X_2{}^{(k)}$ to obtain $X_2{}^{(k+1)}$ $i.e.,$

$$X_2{}^{(k+1)} = X_2{}^{(k)} + S(\tilde{B})\, d^{(k)} \quad (2.4)$$

where $S(\tilde{B})$ is a matrix of order $(n - m) \times m$ whose elements are the signs of the elements of $\tilde{B}^T$ and $d^{(k)}$ is m-dimensional vector whose i^{th} entry is

$$d_i{}^{(k)} = \frac{\tilde{b}_i{}^{(k)} - \tilde{B}_i X_2{}^{(k)}}{m\|\tilde{B}_i\|} \quad (2.5)$$

Step 3: Calculate $\hat{b}^{(k)} = b - \tilde{B}X_2{}^{(k+1)}$ (2.6)

Applying the PAOR method (1.7) with (1.10) or (1.11) or (1.12) whichever is applicable, to the square linear system $By = \hat{b}^{(k)}$ with the initial guess $X_1{}^{(k)}$ to obtain $X_1{}^{(k+1)}$ i. e;

$$X_1{}^{(k+1)} = P_{\alpha,r,\omega} X_1{}^{(k)} + [(1 + \alpha)I - \omega L]^{-1}\, r\hat{b}^{(k)} \quad (2.7)$$

Step 4: Obtain $X^{(k+1)} = \begin{pmatrix} X_1{}^{(k+1)} \\ X_2{}^{(k+1)} \end{pmatrix}$ where $X_2{}^{(k+1)}$ and $X_1{}^{(k+1)}$ are obtained in steps 2 and 3 respectively.

Step 5: If $\|AX^{(k+1)} - b\| < \epsilon$ where ϵ is the fixed threshold, take $X^{(k+1)}$ as the solution of (1.2). If not assign $k \leftarrow k + 1$ and go to step 2.

For $\alpha = 0$ in (2.7), the generalized PAOR method realizes the generalized AOR method and for $\alpha = 0$ and $r = \omega$ the method (2.7) realizes the generalized SOR method discussed by V.B.Kumar Vatti et.al [7].

3. Convergence Criteria:

Let $X^{(k)} = \begin{pmatrix} X_1{}^{(k)} \\ X_2{}^{(k)} \end{pmatrix}$ where $X_1{}^{(k)} \in \mathbb{R}^m$ *and* $X_2{}^{(k)} \in \mathbb{R}^{n-m}$, be the k[th] appropriate solution of the system (1.4). Then,

$$d_i{}^{(k)} = \frac{\tilde{b}_i{}^{(k)} - \tilde{B}_i X_2{}^{(k)}}{m.\|\tilde{B}_i\|}$$

$$= \frac{b_i - B_i X_1{}^{(k)} - \tilde{B}_i X_2{}^{(k)}}{m.\|\tilde{B}_i\|}$$

$$= \frac{b_i - (B_i X_1{}^{(k)} + \tilde{B}_i X_2{}^{(k)})}{m.\|\tilde{B}_i\|}$$

$$= \frac{b_i - A_i X_1{}^{(k)}}{m.\|\tilde{B}_i\|} \qquad (3.1)$$

If we denote the non-singular matrix $\mathrm{diag}(\|\tilde{B}_1\|, \|\tilde{B}_2\|, \ldots.. \|\tilde{B}_m\|)$ by $N(\tilde{B})$, then (3.1) can be put in the form

$$d^{(k)} = \frac{1}{m} N(\tilde{B})^{-1}(b - AX^{(k)}) \qquad (3.2)$$

By the equation (2.4) of step2 of the algorithm, we have $X_2{}^{(k+1)} = X_2{}^{(k)} + \frac{1}{m} S(\tilde{B}).N(\tilde{B})^{-1}(b - AX^{(k)})$ (3.3)

With this the equation (2.6) of step3 further reduces to

$$\hat{b}^{(k)} = b - \tilde{B}X_2{}^{(k)} + \frac{1}{m} S(\tilde{B}).N(\tilde{B})^{-1}(AX^{(k)} - b) \qquad (3.4)$$

The PAOR method now for obtaining $X_1{}^{(k+1)}$ is derived from (2.7) can be rewritten as

$$X_1{}^{(k+1)} = [(1+\alpha)I - \omega L]^{-1}[\{(1+\alpha-r)I + (r-\omega)L + rU\}X_1{}^{(k)} + r\hat{b}^{(k)}]$$

$$= [(1+\alpha)I - \omega L]^{-1}[((1+\alpha)I - \omega L)X_1{}^{(k)} - r\{(I-L-U)X_1{}^{(k)}\}X_1{}^{(k)} + r\hat{b}^{(k)}]$$

$$= X_1{}^{(k)} - r[(1+\alpha)I - \omega L]^{-1}\{BX_1{}^{(k)} - \hat{b}^{(k)}\} \qquad (3.5)$$

Since $AX^{(k+1)} = BX_1{}^{(k+1)} + \tilde{B}X_2{}^{(k+1)}$, from (3.3),(3.4) and (3.5) we can have

$$AX^{(k+1)} = BX_1{}^{(k+1)} + \tilde{B}X_2{}^{(k+1)}$$

$$= BX_1{}^{(k)} - rB\left\{BX_1{}^{(k)} - b + \tilde{B}X_2{}^{(k)} - \tfrac{1}{m}\, S(\tilde{B}).\tilde{B}.\mathrm{N}(\tilde{B})^{-1}(\mathrm{AX^{(k)}} - \mathrm{b})\right\}$$
$$+\tilde{B}X_2{}^{(k)} + \tfrac{1}{m}\, S(\tilde{B}).\tilde{B}.\mathrm{N}(\tilde{B})^{-1}(\mathrm{b} - \mathrm{AX^{(k)}})$$

$$= AX^{(k)} - rB[(1+\alpha)I - \omega L]^{-1}\left\{BX_1{}^{(k)} - b + \tilde{B}X_2{}^{(k)} - \frac{1}{m}\, S(\tilde{B}).\tilde{B}.\mathrm{N}(\tilde{B})^{-1}(\mathrm{AX^{(k)}} - \mathrm{b})\right\}$$
$$+\frac{1}{m}\, S(\tilde{B}).\tilde{B}.\mathrm{N}(\tilde{B})^{-1}(\mathrm{b} - \mathrm{AX^{(k)}})$$

$$= AX^{(k)} - rB[(1+\alpha)I - \omega L]^{-1}\{(\mathrm{AX^{(k)}} - \mathrm{b}) - P(\mathrm{AX^{(k)}} - \mathrm{b})\} - P(\mathrm{AX^{(k)}} - \mathrm{b}) \tag{3.6}$$

where

$$P = \tfrac{1}{m}\,.\tilde{B}.S(\tilde{B})\mathrm{N}(\tilde{B})^{-1} \tag{3.7}$$

$$Now, AX^{(k+1)} - b$$
$$= (AX^{(k)} - b) - rB[(1+\alpha)I - \omega L]^{-1}\{(I-P)(\mathrm{AX^{(k)}} - \mathrm{b})\} - P(\mathrm{AX^{(k)}} - \mathrm{b})$$
$$= (AX^{(k)} - b)[I - rB[(1+\alpha)I - \omega L]^{-1}(I-P) - P]$$
$$= (AX^{(k)} - b)[(I-P) - rB[(1+\alpha)I - \omega L]^{-1}(I-P)] \tag{3.8}$$

From (3.8), we have $X = \lim_k X^{(k)}$ is a solution of the system (1.4) provided

$$\|I - rB[(1+\alpha)I - \omega L]^{-1}\| < 1 \tag{3.9}$$

and $$\|(I-P)\| < 1 \tag{3.10}$$

where P is the matrix as given in (3.7).

For the existence of x, we consider

$$\|X^{(k+1)} - X^{(k)}\| = \|X_1{}^{(k+1)} - X_1{}^{(k)}\| + \|X_2{}^{(k+1)} - X_2{}^{(k)}\|.$$

$$\|r[(1+\alpha)I - \omega L]^{-1}\{(I-P)(\mathrm{b} - \mathrm{AX^{(k)}})\}\| + \left\|\frac{1}{m}\, S(\tilde{B})\mathrm{N}(\tilde{B})^{-1}(\mathrm{b} - \mathrm{AX^{(k)}})\right\|$$
$$\leq \left[\|r[(1+\alpha)I - \omega L]^{-1}(I-P)\| + \tfrac{1}{m}\left\| S(\tilde{B})\mathrm{N}(\tilde{B})^{-1}\right\|\right]\|AX^{(k)} - b\| \tag{3.11}$$

With this we can conclude that $X = \lim_k X^{(k)}$ exists as $X^{(k)}$ is a Cauchy-sequence.

Hence, the generalized PAOR method converges under the conditions (3.9) and (3.10) for any matrix norm.

Remark 3.1: It can be shown that the Generalized AOR and Generalized SOR methods converge under the conditions

$$\|I - rB(I-\omega L)^{-1}\| < 1 \tag{3.12}$$

and

$$\|I - B\| < 1 \tag{3.13}$$

respectively along with the condition (3.10), as done above.

4. Numerical Example:

In this section, we consider the following non-square linear system i.e;

$$\begin{bmatrix} 1 & -2/5 & 0 & -1/5 & 3/5 & -1/5 & 2/5 \\ -14/35 & 1 & -2/7 & 0 & -4/10 & 4/10 & -3/10 \\ 0 & -14/35 & 1 & -1/5 & 3/5 & -3/5 & -1/5 \\ -1/5 & 0 & -1/5 & 1 & -2/5 & 3/5 & -1/5 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} 536/105 \\ -661/210 \\ 64/35 \\ -499/105 \end{pmatrix}$$

This system can be expressed as in (1.5), as

$$\begin{bmatrix} 1 & -2/5 & 0 & -1/5 \\ -14/35 & 1 & -2/5 & 0 \\ 0 & -14/35 & 1 & -1/5 \\ -1/5 & 0 & -1/5 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ p \end{pmatrix} + \begin{bmatrix} 3/5 & -1/5 & 2/5 \\ -4/10 & 4/10 & -3/10 \\ 3/5 & -3/5 & -1/5 \\ -2/5 & 3/5 & -1/5 \end{bmatrix} \begin{pmatrix} q \\ r \\ s \end{pmatrix} = \begin{pmatrix} 536/105 \\ -661/210 \\ 64/35 \\ -499/105 \end{pmatrix}$$

i.e; $BX_1 + \tilde{B}X_2 = b$

It is obtained that the smallest and largest eigenvalues of the corresponding Jacobi matrix of B are $\underline{\mu} = 0,\ \overline{\mu} = 0.595219047.$ Since $\underline{\mu} \neq \overline{\mu}$ and k of (1.13) is less than unity, r and ω of Generalized PAOR method with $\alpha = 0$ are obtained to be $r = 1.143029314$ and $\omega = 1.108915771.$

Applying the procedure given in section 2, we obtained the following data and the non-basic solutions with an error less than 0.5×10^{-10}.

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n	Values of $\|b - AX\|_1$ in successive iterations and obtained final solution upto an error less than 0.5E-10			
	Generalized Jacobi	**Generalized Gauss-Seidal**	**Generalized SOR with** $\omega = 1.108915771$	**Generalized PAOR with** $\alpha = 0$ $r = 1.143029314$ $\omega = 1.108915771$
1	0.73025049	0.60558839	0.7027654408	0.77003132
2	0.10385178	0.1962279	0.2741681878	0.27761455
3	0.01119328	0.008172363419	0.08723062738	0.09555311
. .	. .	. .	. .	. .
18	.	.	.	.
19	.	.	.	1.02001E-8
20	.	.	8.857137468E-9	(x,y,z,p,q,r,s)= (1.50622747, 0.22340117, -1.53509915, -1.17821067 3.08969706, -3.08969706 2.4512379)
. . . . .	. . . . .	. . . . .	(x,y,z,p,q,r,s)= (1.49289105, 0.23640863, -1.53708007, -1.16249469 3.1029273, -3.10292731 2.4789838)	
26		3.123809744E-8		
		(x,y,z,p,q,r,s)= (1.46030331 0.27012639, -1.53341143, -1.12493969, 3.13050381 -3.13050381, 2.55779542)		
37	8.2809477941E-10			
	(x,y,z,p,q,r,s)= (1.58874678 0.16057255, -1.44515934 -1.28318818, 2.96445338 -2.96445338, 2.38010951)			

Table 4.1

5. CONCLUSION

It is clear from the above Table-4.1 that Generalized PAOR method converged a bit faster comparing to the other methods discussed in this paper. And, Generalized AOR method will have the same rate of convergence as that of generalized SOR as $\sqrt{1-\overline{\mu}^2} < 1-\underline{\mu}^2$.

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