SOME PELL SQUARE GRACEFUL GRAPHS

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ABSTRACT

Rosa [9] introduced graceful labeling. Fibonacci graceful labeling was introduced by Acharya and Hegde and further studied in [3]. In [4-5], we introduced and studied Pell graceful labeling. In [7], T. Tharmaraj and P.B. Sarasija introduced the concept of square graceful labeling in 2014 and further studied in [8]. In [6], we introduced a new type of labeling called Pell square graceful labeling of graphs. In this paper , We continue to study its labeling for several graphs.

Keywords: Pell sequence, Pell graceful labeling, Pell graceful graph, Pell square graceful labeling, Pell square graceful graph.

AMS(MOS) Subject Classification: 05C78

1. INTRODUCTION

All graphs considered here are simple, finite and undirected. The terms not defined in this paper are used as in Harary [2]. This paper deals with the research on graph labeling. Labeled graphs find their applications in Coding Theory and Communication Network Addressing.

Rosa introduced the concept of graceful labeling f of a (p, q) graph G as follows: f is a graceful labeling if f is an injection from $V(G)$ to the set

$\{0, 1, 2, ..., q\}$ such that when each edge uv is assigned the label $|f(u)-f(v)|$, the resulting edge labels are distinct.

By following Acharya and Hegde, a new type of labeling called Fibonacci graceful labeling is introduced. The numbers $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, ...$ is the sequence of Fibonacci numbers.

As an extension to Fibonacci graceful labeling, in [4-5], we introduced Pell graceful labeling of graphs and studied its labeling for varieties of graphs. In [7], T. Tharmaraj and P.B. Sarasija introduced the concept of square graceful labeling in 2014 and further studied in [8]. In this paper, we introduce a new type of labeling called Pell square graceful labeling of graphs. Here, we investigate the family of graphs which admits Pell square graceful labeling.

2. PRIOR RESULTS

Definition 2.1

Let $G(p, q)$ be a graph. A injective function f from $V(G)$ into $\{0, 1, 2, ..., F_q\}$, where F_q is the q^{th} Fibonacci number is said to be Fibonacci graceful if the induced edge labeling $f^*(uv) = |f(u)-f(v)|$ is a bijection onto the set $\{F_1, F_2, ..., F_q\}$.

If a graph $G(p, q)$ admits a **Fibonacci graceful labeling** then G is called a **Fibonacci graceful graph**.

As an extension to Fibonacci graceful labeling, we introduce Pell graceful labeling.

Definition 2.2

Let $G(p, q)$ be a graph. An injective function f from $V(G)$ into $\{0, 1, 2, ..., p_q\}$ where p_q is the q^{th} Pell number in the Pell sequence is said to be **Pell graceful** if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{p_1, p_2, ..., p_q\}$.

If a graph $G(p, q)$ admits a Pell graceful labeling then G is called a **Pell graceful graph.**

Remark 2.3

The classic Pell sequence is obtained as follows:

$P_0 = 0$, $P_1 = 1$ and $P_{n+1} = 2P_n + P_{n-1}$ for all $n \geq 1$.

(i.e.) {0, 1, 2, 5, 12, 29, 70, ...} is the Pell sequence.

Illustration 2.4

In Fig. 1, we provide an example of a Pell graceful labeling of a graph.

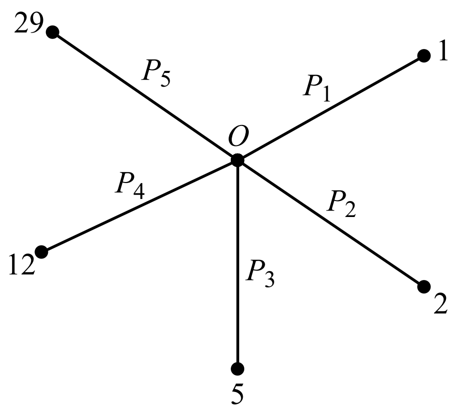

Fig. 1

Definition 2.5

Let $G(p, q)$ be a graph. An injective function f from $V(G)$ into $\{0, 1, 2, ..., p_q^2\}$ where p_q is the q^{th} Pell number in the Pell sequence is said to be **Pell square graceful labeling** (PSGL) if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{p_1^2, p_2^2, ..., p_q^2\}$.

The graph which admits Pell square graceful labeling is called **Pell square graceful graph** (PSGG).

Theorem 2.6 : The path P_n is Pell square graceful for all $n \geq 3$.

Theorem 2.7 : The star $K_{1,n}$ is Pell square graceful for all $n \geq 3$.

Theorem 2.8 : The Bistar $B_{m,n}$ is a Pell square graceful graph for all m , $n \geq 2$.

Theorem 2.9 : The graph (P_n, S_m) is Pell square graceful for all $n \geq 3$, $m \geq 2$.

Theorem 2.10 : The graph obtained by the subdivision of the edges of the star $K_{1,n}$ is Pell square graceful graph for all n.

Theorem 2.11: The graph obtained by the subdivision of the central edge of the bistar $B_{m,n}$ is Pell square graceful for all m and $n \geq 3$.

3. MAIN RESULTS

Theorem 3.1

The graph obtained by the subdivision of the edges of stars of $B_{m,n}$ is a Pell square graceful graph.

Proof

Let $B_{m,n}$ be a star with $m + n + 2$ vertices and $m + n + 1$ edges.

Let $SE(B_{m,n})$ denote the subdivision of edges of $B_{m,n}$.

Let $G = SE(B_{m,n})$

Let $V(G) = \{u, v, u_i, 1 \le i \le m, v_i, 1 \le i \le m, u_i^{'}, 1 \le i \le n, u_i^{'}, 1 \le i \le n\}$ be the set of vertices and $E(G) = \{e = (uv), e_i, 1 \le i \le m, f_i, 1 \le i \le m, e_i^{'}, 1 \le i \le n, f_i^{'}, 1 \le i \le n\}$ be the set of edges where $e = uv$, $e_i = uu_i$, $f_i = u_iv_i$, $e_i^{'} = vu_i^{'}$, $f_i^{'} = u_i^{'}v_i^{'}$ as denoted in Fig. 2. Here $q = 2m + 2n + 1$.

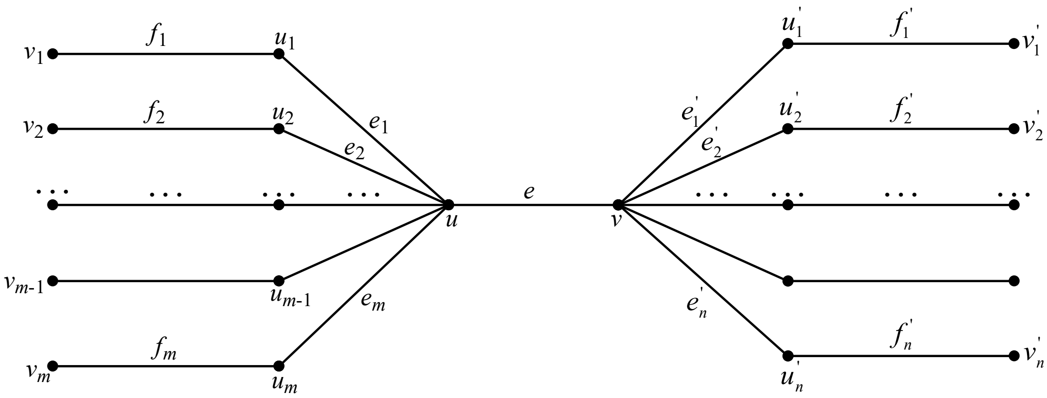

Fig. 2: Ordinary labeling of *SE*($B_{m,n}$)

We first label the vertices as follows.

Define $f : V(G) \to \{0, 1, 2, ..., p_q^2\}$ by

$$f(u) = 0$$

$$f(v) = 1$$

For $1 \le i \le m$,

$$f(u_i) = p_{m+2n+1+i}^2$$

For $1 \le i \le m$,

$$f(v_i) = p_{m+2n+1+i}^2 - p_{i+1}^2$$

For $1 \le i \le n$,

$$f(u_i') = p^2_{m+n+1+i} + 1$$

For $1 \le i \le n$,

$$f(v_i') = p^2_{m+n+1+i} + 1 - p^2_{m+1+i}$$

The above defined function induces a bijection $f^* : E(SE(B_{m,n})) \to \{p_1^2, p_2^2, ..., p_q^2\}$. Hence the graph $SE(B_{m,n})$ is a Pell square graceful graph for all $m, n \ge 2$.

Example 3.2

PSGL of $(SE(B_{3,2}))$ is given in Fig. 3.

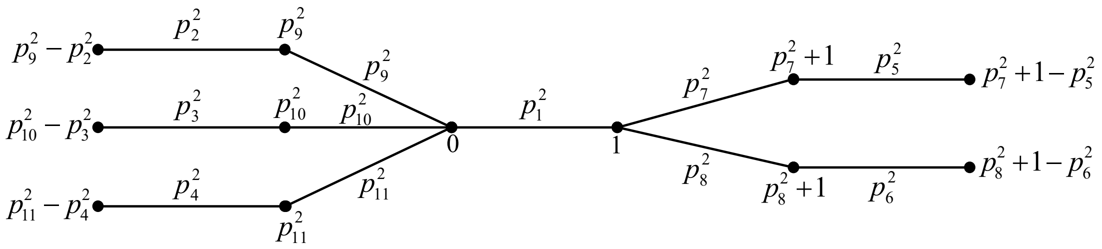

Fig. 3: PSGL of (*SE*(*B*$_{3,2}$))

Example 3.3

PSGL of $(SE(B_{4,4}))$ is given in Fig. 4.

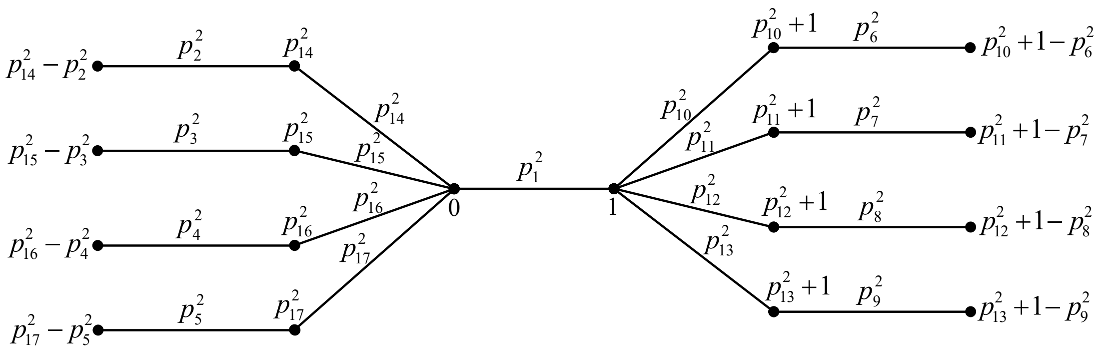

Fig. 4: PSGL of (*SE*(*B*$_{4,4}$))

Example 3.4

PSGL of ($SE(B_{3,6})$) is given in Fig. 5.

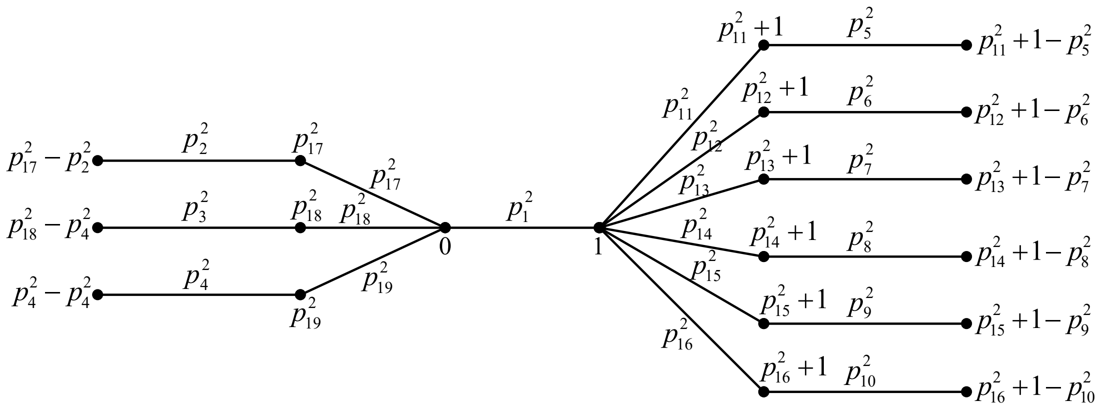

Fig. 5: PSGL of ($SE(B_{3,6})$)

Theorem 3.5

The graph obtained by the subdivision of central edge of the graph $SE(B_{m,n})$ is Pell square graceful for all $m, n \geq 2$.

Proof

Let the vertices and edges of the subdivision of central edge of $SE(B_{m,n})$ be as given in Fig. 6.

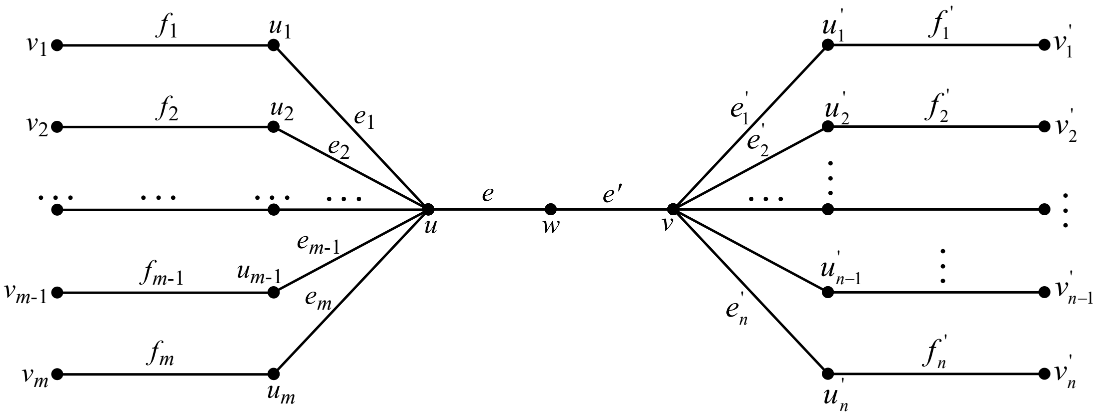

Fig. 6: Ordinary labeling of $G = S(CE(SE(B_{m,n})))$

We first label the vertices as follows.

Define f: $V(G) \to \{0, 1, 2, ..., p_q^2\}$ by

$$f(u) = 0; \quad f(w) = 1; \quad f(v) = 5$$

For $1 \le i \le m$, $\quad f(u_i) = p^2_{m+2n+2+i}$

For $1 \le i \le m$, $\quad f(v_i) = p^2_{m+2n+2+i} - p^2_{i+2}$

For $1 \le i \le n$, $\quad f(u_i') = p^2_{m+n+2+i} + 5$

For $1 \le i \le n$,

$$f(v_i') = p^2_{m+n+2+i} + 5 - p^2_{m+2+i}$$

The above defined function induces a bijection $f^*: E(S(CE(SE(B_{m,n})))) \to \{p_1^2, p_2^2, ..., p_q^2\}$. Therefore, the graph considered is a Pell square graceful graph for all $m, n \ge 2$.

Example 3.6

PSGL of $S(CE(SE(B_{9,4})))$ is given in Fig. 7.

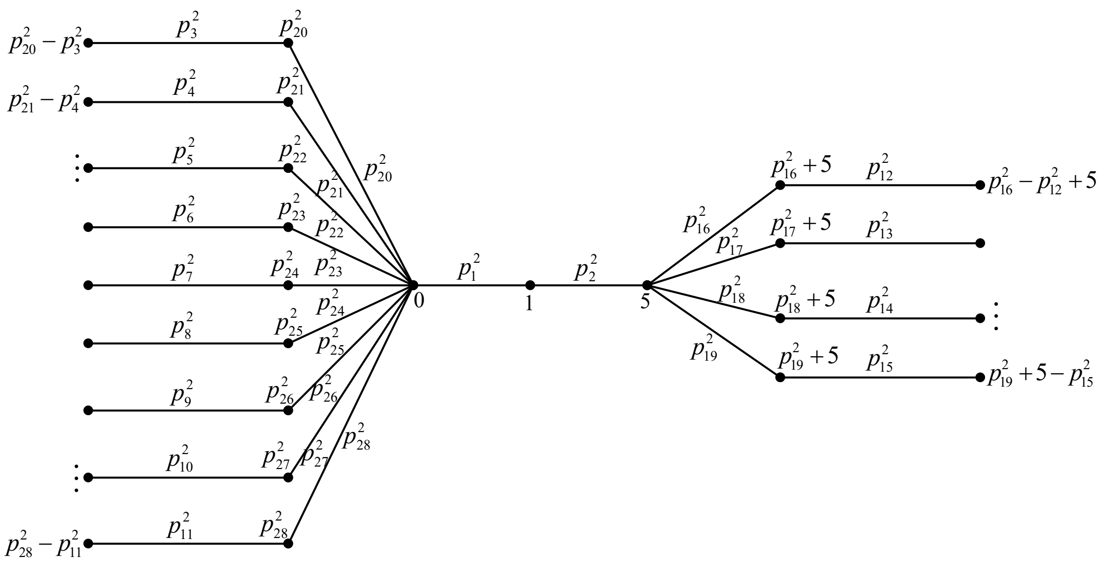

Fig. 7: PSGL of $S(CE(SE(B_{9,4})))$

Remark 3.7

The graph $C_3 \ominus K_{1,4}$ is not Pell square graceful graph.

Theorem 3.8

The graph $P_m \odot nK_1$ is a Pell square graceful graph for all $m \geq 3$ & $n \geq 2$.

Proof

Let $V(P_m \odot nK_1) = \{u_1, u_2, ..., u_m, u_{11}, u_{12}, ..., u_{1n}, u_{21}, u_{22}, ..., u_{2n}, ..., u_{m1}, u_{m2}, ..., u_{mn}\}$ and edges are denoted as given in Fig. 8. Here $q = (mn + m - 1)$.

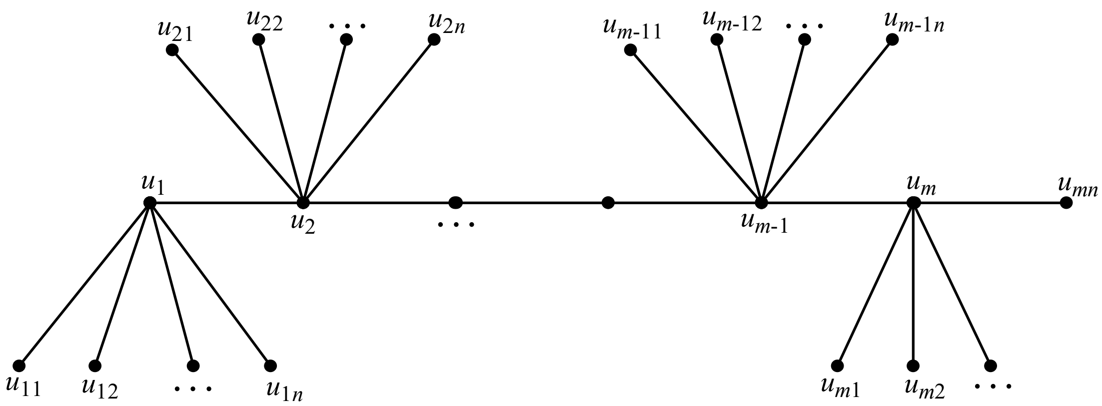

Fig. 8: Ordinary labeling of $P_m \odot nK_1$

Define a function $f : V(P_m \odot nK_1) \rightarrow \{0, 1, 2, ..., p_q^2\}$ by

$$f(u_1) = 0 ; \quad f(u_i) = \sum_{t=1}^{i-1} p_t^2$$

For $1 \leq i \leq m$, $1 \leq j \leq n$,

$$f(v_{ij}) = p^2_{(2m-i)-m(j-n)} - f(u_i)$$

The above defined function induces a bijection $f^*: E(P_m \odot nK_1) \rightarrow \{p_1^2, p_2^2, ..., p_q^2\}$. Hence the graph $P_m \odot nK_1$ is a Pell square graceful graph for all $m \geq 3, n \geq 2$.

Example 3.9

PSGL of $P_5 \odot 4K_1$ is given in Fig. 9.

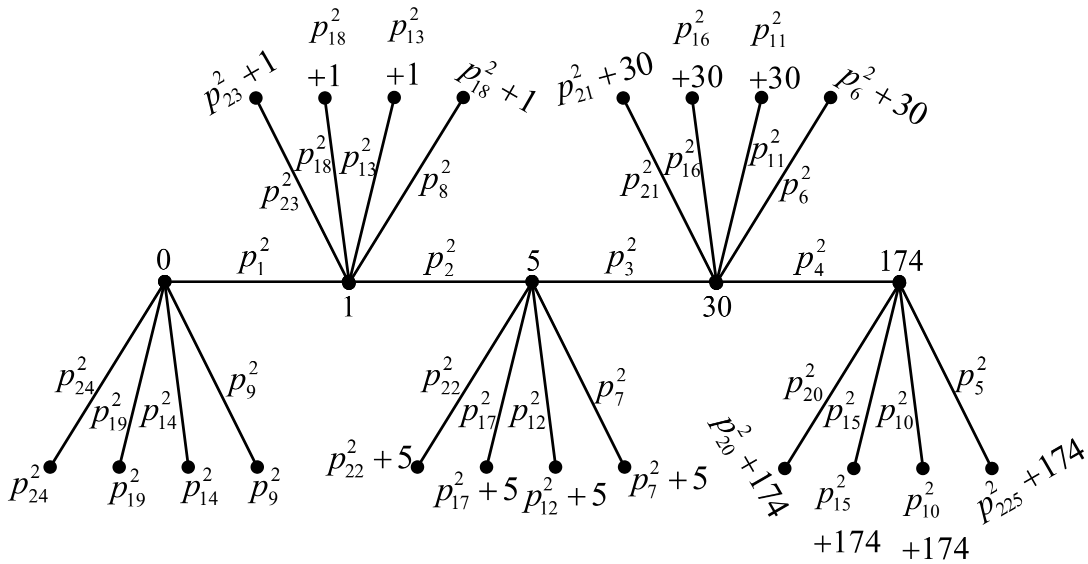

Fig. 9: PSGL of $P_5 \odot 4K_1$

Theorem 3.10

The Comb $P_n \odot K_1$ is a Pell square graceful graph for all n.

Proof

By taking $m = n$ & $n = 1$ in Theorem 3.9 we get labeling of $P_n \odot K_1$.

Example 3.11

PSGL of $P_5 \odot K_1$ is given in Fig. 10.

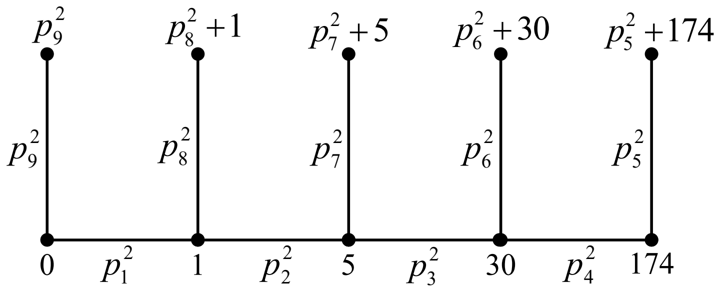

Fig. 10: PSGL of $P_5 \odot K_1$

Theorem 3.12

The graph obtained by the subdivision of the edges of the path P_n in $P_n \odot K_1$ is a Pell square graceful graph for $n \geq 3$.

Proof

Let $V(P_n \odot K_1) = \{u_i, 1 \leq i \leq n\} \cup \{v_i, 1 \leq i \leq n\} \cup \{w_i, 1 \leq i \leq n-1\}$ be the set of vertices and $E(P_n \odot K_1) = \{u_iw_i, 1 \leq i \leq n-1\} \cup \{w_iu_{i+1}; 1 \leq i \leq n-1\} \cup \{u_iv_i; 1 \leq i \leq n\}$ be the set of edges as shown in Fig. 11. Here $q = 3n - 2$.

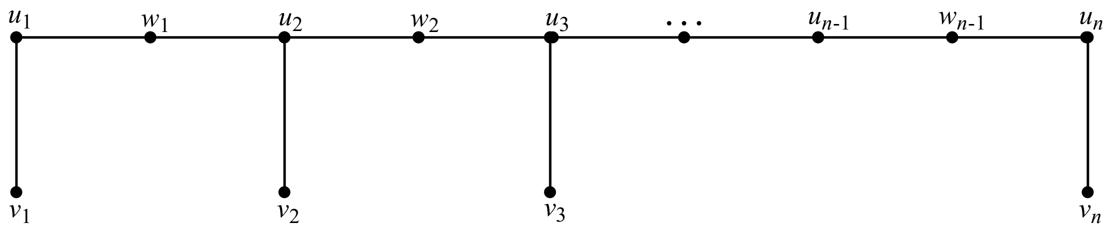

Fig. 11: Ordinary labeling of $P_n \odot K_1$

We first label the vertices as follows.

Define a function $f: V(P_n \odot K_1) \rightarrow \{0, 1, 2, 3, ..., q^2\}$ by

$$f(u_1) = p_q^2$$

$$f(u_2) = 0$$

For $3 \leq i \leq n$,

$$f(u_i) = \sum_{t=1}^{i-2} p_t^2$$

For $2 \le i \le n$,

$$f(v_i) = p_{3n-i-2}^2 + f(u_i)$$

$$f(v_1) = p_q^2 - p_{q-1}^2$$

The above defined function induces a bijection $f^* : E(P_n \odot K_1) \to \{p_1^2, p_2^2, ..., p_q^2\}$. Hence $P_n \odot K_1$ is Pell square graceful graph.

Example 3.13

PSGL of $P_5 \odot K_1$ is shown in Fig. 4.12.

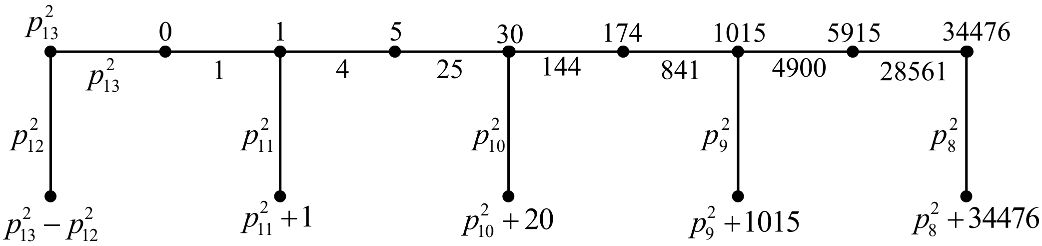

Fig. 12: PSGL of $P_5 \odot K_1$

Theorem 3.14

The graph $\langle S_n : m \rangle$ is a Pell square graceful graph.

Proof

Let the vertices of $\langle S_n : m \rangle$ be $\{u, v_i; 1 \le i \le n, v_{ij}; 1 \le j \le m\}$.

Let the edges of $\langle S_n : m \rangle$ be as denoted in Fig. 13. Here $q = mn + n$.

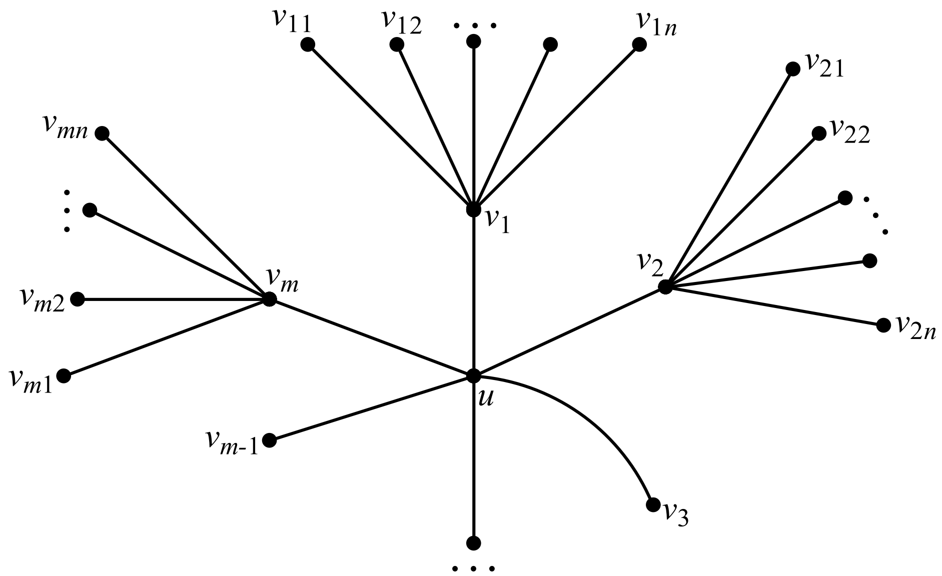

Fig. 13: Ordinary labeling of $\langle S_n : m\rangle$

First we label the vertices as follows.

Define f: $\left(V\left\langle S_n : m\right\rangle\right) \rightarrow \{0, 1, 2, ..., p_q^2\}$ by

$$f(u) = 1$$

$$f(v_i) = p_{m-i+1}^2 + 1 \qquad \text{for } 1 \le i \le m-1$$

$$f(v_m) = 0$$

For $1 \le i \le m$, $1 \le j \le n$,

$$f(v_{ij}) = p_{(m-n)+ni+j}^2 + f(v_i)$$

The above defined function induces a bijection from $f^* : E\left\langle S_n : m\right\rangle \rightarrow \left\{p_1^2, p_2^2, ..., p_q^2\right\}$. Hence the graph $\langle S_n : m\rangle$ is Pell square graceful graph for all n, $m \ge 2$.

Example 3.15

PSGL of $\langle S_3 : 3\rangle$ is shown below in Fig. 14.

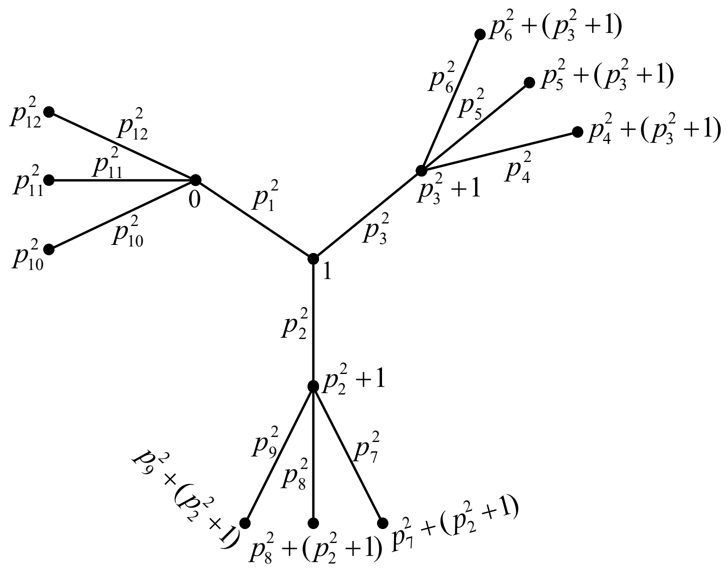

Fig. 4.14: PSGL of $\langle S_3 : 3\rangle$

Theorem 3.16

Olive tree O_n is Pell graceful for all $n \geq 3$.

Proof

Let O_n be the Olive tree having n paths of length 1, 2, ..., n adjoined at one vertex v_0.

Let the vertices of O_n be $\{v_0, v_{11}, v_{12}, ..., v_{1n}, v_{21}, v_{22}, ..., v_{2(n-1)}, v_{31}, v_{32}, ..., v_{3(n-2)}, ..., v_{n(1)}\}$. The ordinary labeling of O_n be as given in Fig. 15.

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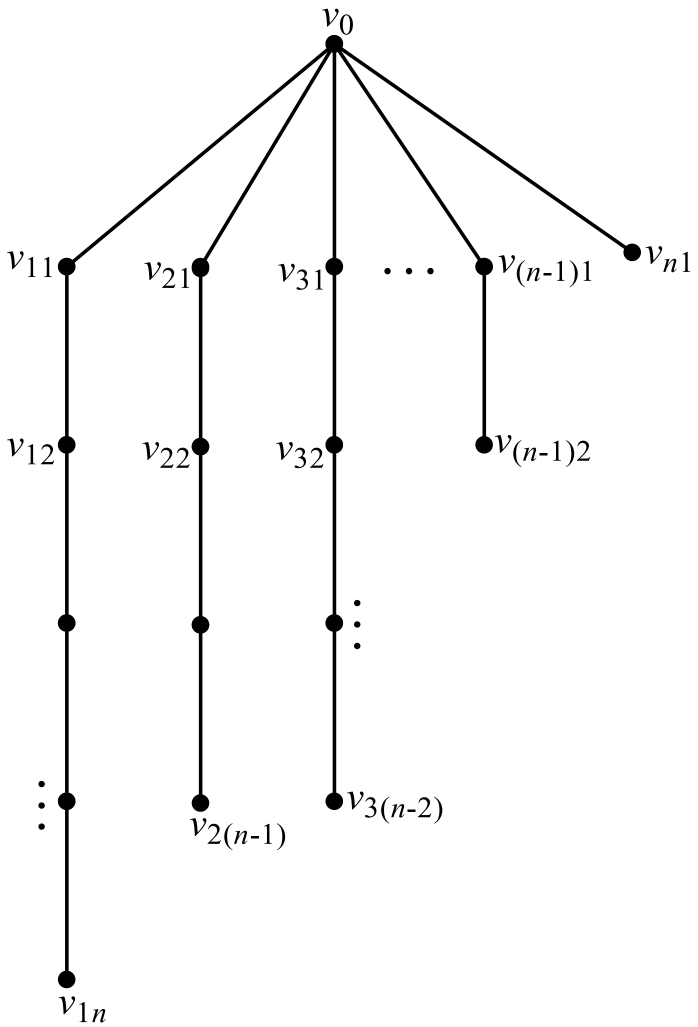

Fig. 15: Ordinary labeling of olive tree O_n

We first label the vertices as follows.

Define $f: V(O_n) \to \{0, 1, 2, ..., p_q^2\}$ by

$$f(v_0) = 0$$

$$f(v_{i1}) = p_{q+1-i}^2 \qquad i = 1, 2, ..., n$$

$$f(v_{i2}) = f(v_{i1}) - p_{m+1-n-i}^2 \qquad i = 1, 2, ..., n-1$$

$$f(v_{i3}) = f(v_{i2}) - p_{m+2-2n-i} \qquad i = 1, 2, ..., n-2$$

Continuing in this manner, we get labels for v_{i4}, v_{i5}, ...

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The above defined function induces a bijection $f^*: E(O_n) \rightarrow \{p_1^2, p_2^2, ..., p_q^2\}$. Hence Olive tree O_n is Pell square graceful.

Example 3.17

The PSGL of O_3 is given in Fig. 16.

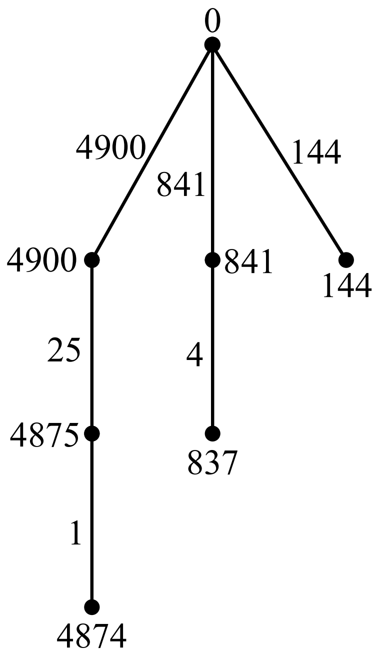

Fig. 16: PSGL of Olive tree O_3

Example 3.18

The PSGL of O_4 is given in Fig. 17.

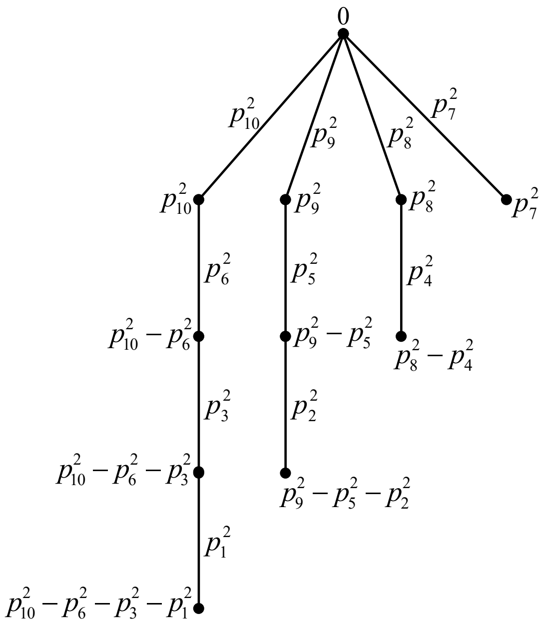

Fig. 17: PSGL of Olive tree O_4

Theorem 3.19

The Twig graph TW_n is Pell square graceful for all $n \geq 3$.

Proof

Let $V(TW_n) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n, v_1', v_2', ..., v_n'\}$ and $E(TW_n) = \{e_1, e_2, ..., e_{n-1}, f_1, f_2, ..., f_n, g_1, g_2, ..., g_n\}$.

Ordinary labeling of T_n is shown in Fig. 18.

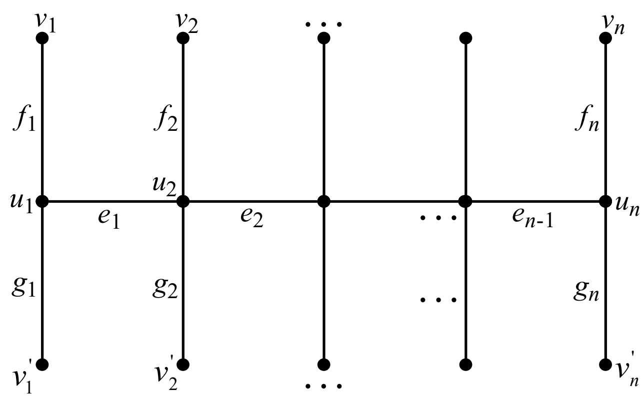

Fig. 18: Ordinary labeling of TW_n

We first label the vertices as follows.

Define $f: V(TW_n) \to \{0, 1, 2, ..., p_q^2\}$ by

$$f(u_1) = 0,$$

For $2 \le i \le n,\ f(u_i) = \sum_{t=1}^{i-1} p_t^2$

For $1 \le i \le n$,

$$f(v_i') = p_{n+i-1}^2 + f(u_i)$$

For $1 \le i \le n$,

$$f(v_i) = p_{q+1-i}^2 + f(u_i)$$

The above defined function induces a bijection. Therefore TW_n is a Pell square graceful graph for all $n \ge 3$.

Example 3.20

The PSGL of Twig TW_5 is shown in Fig. 19.

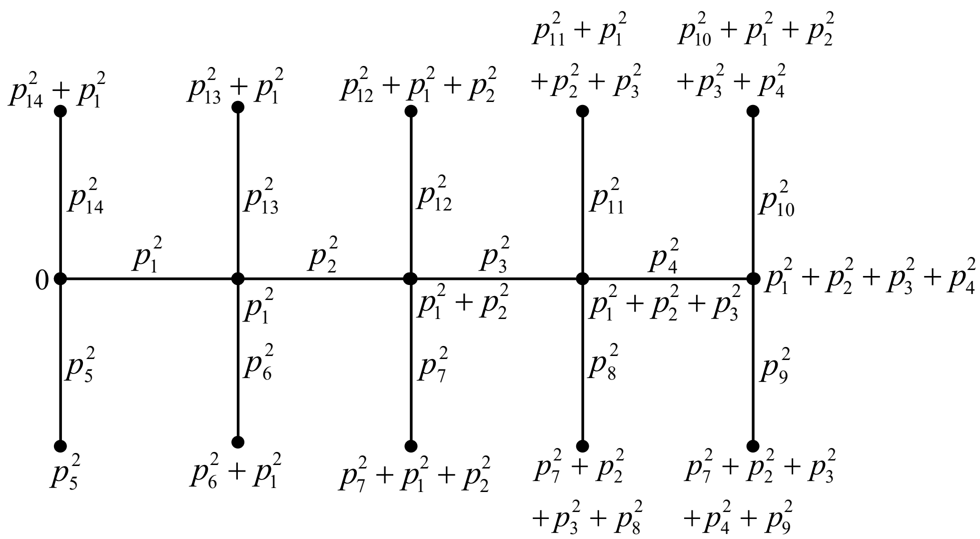

Fig. 19: PSGL of *TW*₅

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