EVALUATION OF DEFINITE INTEGRALS USING BETA AND GAMMA FUNCTIONS

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Abstract:

In this paper we can study about the introduction of special functions like Beta and Gamma functions and the properties..The main aim of this paper is to evaluate the fractional integral and fractional derivative using general method and by using the special functions Beta and Gamma.

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Key words: *Betafunction, Gamma function and fractional calculus.*

Introduction:

In this paper, we are going to discuss the definition, properties and evaluation of integration using beta and gamma functions.The beta functionplays a key role in the calculus as it has a close connection with the gamma function, also known as Euler's integral of the first kind. Which itself works as the generalization of the factorial function. The Beta function was first studied by Euler and Legendre and was specified its name by Jacques Binet.

The gamma function is a continuous expansion to the factorial function, which is only defined for the non-negative integers. Although there are other continuous extensions to the factorial function.The gamma and beta functions are also used for the simplification of various integrals and in the definition of other special functions, such as the Bessel, Legendre functions.

Fractional calculus is a field of mathematical study that grows out of the traditional definitions of calculus integral and derivative operators in to a great extent the same way fractional exponents isconsequence of exponents with integer valuemany researchers found, using their own notation and method, definitions that fit the concept of a non-integer order integral or derivative. Understanding of definitions and exploit of fractional calculus requires the profound knowledge of Gamma function, Beta function.

1. INTRODUCTION OF BETA FUNCTION:

The beta function is define as the domains ofreal numbers. The notation tosymbolize the beta function is "β". The beta function is meant by B (m,n), where the parameters m and n should be real numbers

DEFINITION :

If m> 0, n > 0, then Beta function is defined by the integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ and is denoted by β(m, n)

i. e. $\mathbf{B(m, n)} = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ **……………………….(1)**

Properties of Beta function:

1.1 Beta function is a symmetric function. B (m, n) = B (n, m), where m > 0, n >0

1.2. $B(m, n) = \int_0^{\frac{\pi}{2}} sin^{2m-1}\theta cos^{2n-1}\theta \, d\theta$

1.3. $\int_0^{\frac{\pi}{2}} sin^{p}\theta cos^{q}\theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

1.4. B (m, n) =B(m+1,n)+B(m,n+1)

1.5. $B(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$

2. INTRODUCTION OF GAMMA FUNCTION:

The notation to signify the Gamma function is "**Γ**". The gamma function is meant by Γ(n), where the parameter n should be real number

DEFINITION:

If n > 0, then Gamma function is defined by the integral $\int_0^\infty e^{-x}x^{n-1}\,dx$ and is denoted by $\Gamma(n)$

I.e. $\mathbf{\Gamma(n)} = \int_0^\infty e^{-x}x^{n-1}dx$ **……………………………… (2)**

A simple introduction to the renowned gamma function Γ(n) and its various representations. Some of its most important properties are explained.

Properties of Gamma function:

2.1. Reduction formula for Gamma Function $\Gamma(n+1) = n\Gamma(n)$; where n >0.

2.2.If n is a positive integer, then $\Gamma(n+1) = n!$ (n=0, 1, 2, 3…)

2.3.If n is a positive fraction, then $\Gamma(n) = (n-1)(n-2)\ldots(n-r)\Gamma(n-r)$ where (n-r)>0

2.4.Second Form of Gamma Function $\int_0^\infty e^{-x^2} x^{2m-1} dx = \frac{1}{2}\Gamma(m)$

2.5. $\int_0^{\frac{\pi}{2}} sin^p\theta cos^q\theta \; d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$

2.6. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

2.7. $\Gamma\left(\frac{n+1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{n!4^n}$ for n=0,1,2,3………so on

3. Realtion between gamma and beta function:

$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ where m>0, n>0 ………………………. (3)

We revise the gamma and beta functions of complex variable for the development of fractional calculus.

Evauation of fractional integral using general method:

Example 1. Evaluate $\int_0^\infty x^3 e^{-x} dx$

Solution (A1): using the general integration

i.e. by parts we have

$\int_0^\infty x^3 e^{-x} dx = [x^3(-e^{-x}) - 3x^2(e^{-x}) + 6x(-e^{-x}) - 6(e^{-x}) + 0]_0^\infty$

$=[-e^{-x}(x^3+3x^2+6x+6)]_0^\infty$

$= -(0-6)$

$\therefore \int_0^\infty x^3 e^{-x}dx = 6$ ……………………….(4)

Evauation of fractional integral by using Gamma function:

Solution (B1): $\int_0^\infty x^3 e^{-x}dx$

From Eq.(1) we have, $\int_0^\infty e^{-x}x^{4-1}\,dx = \Gamma(4)$

From property no. 2.2 we have, $\Gamma(3+1) = 3! = 6$

$\therefore \int_0^\infty x^3 e^{-x}dx = 6$ ………………………… (5)

The result of example 2 from equation (4) and (5) are same

From Example 1 we can conclude that solution (A1)is long process as compare to solution (B1)

Example 2. Evaluate $\int_0^{\frac{\pi}{2}} sin^2\theta cos^2\theta d\theta$

Solution (A2): using the general integration

$\int_0^{\frac{\pi}{2}} sin^2\theta cos^2\theta\ d\theta = \int_0^{\frac{\pi}{2}}\left(\frac{1-cos2\theta}{2}\right)\left(\frac{1+cos2\theta}{2}\right)d\theta$

$=\frac{1}{4}\int_0^{\frac{\pi}{2}}\left[1-\left(\frac{1+cos4\theta}{2}\right)\right]d\theta$

$=\frac{1}{4}\int_0^{\frac{\pi}{0}}\left(\frac{1-cos4\theta}{2}\right)d\theta$

$=\frac{1}{8}\int_0^{\frac{\pi}{2}}d\theta-\frac{1}{8}\int_0^{\frac{\pi}{2}}cos4\theta d\theta$

$=\left[\frac{\theta}{8}\right]_0^{\frac{\pi}{2}}-\left[\frac{sin4\theta}{32}\right]_0^{\frac{\pi}{2}}$

$=\frac{\pi}{16}-0$

$\therefore \int_0^{\frac{\pi}{2}} sin^2\theta cos^2\,\theta \;\; d\theta \;=\frac{\pi}{16}$................................ (6)

Evauation of fractional integral using Gamma and Beta function:

Solution (B2): From property no.1.3we have,

$\int_0^{\frac{\pi}{2}} sin^2\theta cos^2\,\theta \;\; d\theta=\frac{1}{2}B\left(\frac{2+1}{2},\frac{2+1}{2}\right)$

From eq. (3) wehave,

$\frac{1}{2}B\left(\frac{3}{2},\frac{3}{2}\right)=\frac{1}{2}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{3}{2}\right)}$

From property no.2.6 and property no 2.2 we have,

$\frac{1}{2}\frac{\left(\frac{1}{2}\right)\sqrt{\pi}\left(\frac{1}{2}\right)\sqrt{\pi}}{\Gamma(3)} \;\; =\frac{\pi}{16}$...................... (7)

The result of example 1 from equation (6) and (7) are same

From the above Example 2 we can conclude that solution (B2) is long process as compare to solution (A2)

Conclusion:

Here we can conclude how the properties of these functions may be used for integral evaluation. In addition, it demonstrates how these functions may be used to remove the long and tedious traditional methods of integral evaluation for many integrals. Evaluation of integrals by using general integration consumes more time than using beta and gamma functions.

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